Algebraic Geometry and Analytic Geometry (GAGA)

JP Serre

tr. T Waring

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Notes

The world probably doesn't need another English translation of GAGA, but I need an activity so here we are. This will, most likely, end up relatively rough, but should be (I hope!) correct. As such, do let me know if anything is unclear or incorrect. I am working from the French available here [Ser56].

I will give the references as in the original, supplemented occaisionally by a modern treatment, if I have come across one. Principally this will be from the excellent [GLS07], on the theory of analytic spaces. The principal references in the original are to Cartan's *Séminaire E.N.S.*, all volumes of which are available here.

After this section I will put anything that is me (ie: not Serre) in a footnote.

Introduction

Let X be a projective algebraic variety, defined over the complex numbers. We can study X from two points of view: the *algebraic* point of view, where the objects of interest are the local rings at points of X, and rational or regular mappings from X to other varieties; and the *analytic* point of view (sometimes called "transcendent") in which holomorphic functions on X play the principal role. We know that the second point of view is particularly fertile if X is non-singular, allowing us to apply techniques from the theory of Kähler manifolds.

For a number of questions, the two points of view give us essentially equivalent results, although the methods are different. For example, we know that globally defined holomorphic differential forms are exactly those rational differential forms of the first kind (supposing still that X is non-singular). Chow's theorem is another example of the same type: all closed analytic subspaces of X are algebraic varieties.

The principal goal of this paper is to understand this equivalence in terms of *coherent sheaves*. More precisely, we demonstrate that coherent algebraic sheaves correspond bijectively to coherent analytic sheaves, and that the equivalence (**??**) between the two categories of sheaves gives us an isomorphism of their cohomology groups (see Section 3.4 for the statements). We will indicate diverse applications of these results, notably to the comparison between analytic and algebraic fibre bundles.

The first two sections are preliminaries. In Section 1 we give the definition and principal properties of *analytic spaces*. The definition we adopt was proposed by Cartan in [Car], thought we drop his restriction to normal varieties. A similar definition is used by Chow in his, as yet unpublished, work on this subject. In Section 2, we associate to any algebraic variety X the structure of an analytic space, and derive its elementary properties. Without doubt the most important of these is that, if $\mathcal{O}_x$ (resp. $\mathcal{H}_x$) denotes the local ring (resp. ring of germs of holomorphic functions) of X at a point x, then the rings $\mathcal{O}_x$ and $\mathcal{H}_x$ form a flat pair[1].

[1]Two rings $A \subset B$ are called a flat pair if B/A is a flat A-module.

Section 3 contains the proofs of the results alluded to above. These proofs rest principally on the theory of coherent algebraic sheaves developed in [Ser55b], and on Cartan's theorems A and B [Car, 1953-4, §18-19]. To be complete, we will reproduce the proofs of these theorems.

Section 4 is dedicated to applications: invariance of Betti numbers by automorphisms of $\mathbb{C}$, Chow's theorem, and comparison of the analytic and algebraic fibre bundles of the structure group of a given algebraic group. Our results on the latter question are incomplete: among semi-simple groups we have only treated symplectic and unimodular (linear) groups.

Finally, we include in Appendix A certain results on local rings which could not be found explicitly in the literature.

1 Analytic spaces

1.1 Analytic subsets of affine space

Let $n \geq 0$ be an integer, and give $\mathbb{C}^n$ its usual topology. For a subset $U \subset \mathbb{C}^n$, we say that U is *analytic* if, for every $x \in U$, there are functions $f_1, \dots, f_k$ holomorphic on an open neighbourhood W of x, so that:

$$U \cap W = \{z \in W \mid f_1(z) = \cdots = f_k(z) = 0\}.$$

Any analytic subset is locally closed in $\mathbb{C}^n$ (the intersection of an open and a closed set), and as such locally compact with the induced topology.

We now assign a sheaf to the topological space U. For any topological space X, let $\mathcal{C}(X)$ be the sheaf of germs of complex-valued functions on X. If $\mathcal{H}$ denotes the sheaf holomorphic functions on $\mathbb{C}^n$, then $\mathcal{H}$ is a subsheaf of $\mathcal{C}(\mathbb{C}^n)$. Given a point x of U, we have a restriction homomorphism:

$$\epsilon_x : \mathcal{C}(\mathbb{C}^n)_x \longrightarrow \mathcal{C}(U)_x.$$

The image of $\mathcal{H}_x$ under ϵ_x is a sub-ring $\mathcal{H}_{x,U}$ of $\mathcal{C}(U)_x$; and the rings $\mathcal{H}_{x,U}$ form a sub-sheaf $\mathcal{H}_U$ of $\mathcal{C}(U)$, which we call the sheaf of holomorphic functions on U. We denote by $\mathcal{A}_x(U)$ the kernel of $\epsilon_x : \mathcal{H}_x \to \mathcal{H}_{x,U}$. By definition:

$$\mathcal{A}_x(U) = \{f \in \mathcal{H}_x \mid f|_{W \cap U} = 0,\, x \in W \text{ open}\}.$$

We frequently identify $\mathcal{H}_{x,U}$ with the quotient $\mathcal{H}_x / \mathcal{A}_x(U)$.

With a topology and a sheaf of functions, we can define the notion of a holomorphic mapping (cf [Car, exp. 6] and [Ser55b, 32]).

Let U and V be analytic subsets of $\mathbb{C}^r$ and $\mathbb{C}^s$ respectively. A mapping $\varphi : U \to V$ is called holomorphic if it is continuous, and if for every $f \in \mathcal{H}_{\varphi(x),V}$, we have $f \circ \varphi \in \mathcal{H}_{x,U}$. This is equivalent to each of the s components of φ being holomorphic functions of $x \in U$.

The composite of two holomorphic mappings is holomorphic. A bijection $\varphi : U \to V$ is called an *analytic isomorphism* (or simply an isomorphism) if φ and φ^{-1} are holomorphic; this is equivalent to the statement that φ is a homeomorphism $U \to V$ which induces an isomorphism between the sheaves $\mathcal{H}_U$ and $\mathcal{H}_V$.

If U and U' are two analytic subsets of $\mathbb{C}^r$ and $\mathbb{C}^{r'}$, the product $U \times U'$ is an analytic subset of $\mathbb{C}^{r+r'}$. The properties laid out in [Ser55b, 33] carry over to this situation, replacing everywhere "locally closed subset" with "analytic subset", and "regular function" with "holomorphic function". In particular, if $\varphi : U \to V$ and $\varphi' : U' \to V'$ are analytic isomorphisms, then so is

$$\varphi \times \varphi' : U \times U' \longrightarrow V \times V'.$$

However, unlike the algebraic case, the topology of $U \times U'$ is the product of the topologies of U and U'.

1.2 The notion of analytic space

Definition 1. An analytic space is a topological space X, and a subsheaf $\mathcal{H}_X$ of $\mathcal{C}(X)$ satisfying the following axioms.

(H1) There is an open cover $\{V_i\}$ of X, such that V_i — with its induced topology and sheaf — is isomorphic to an analytic subset U_i of some affine space.

(H2) The topology on X is Hausdorff.

The definitions of the previous subsection are local, so apply equally to analytic spaces. As such, we refer to $\mathcal{H}_X$ as the sheaf of holomorphic functions on the analytic space X. Defining holomorphic mappings $\varphi : X \to Y$ in the same way, we obtain a family of *morphisms*[2] (in the sense of Bourbaki) for the structure of an analytic space.

If V is an open subset of an analytic space X, a *chart* on V is an isomorphism from V to some analytic subset U. The axiom (H2) indicates that it is possible to recover X from the open sets possessing charts. A subset Y of X is said to be analytic if, for every chart $\varphi : V \to U$ the image $\varphi(Y \cap V)$ is an analytic subset of U. Any such Y is locally closed in X, and can be given the structure of an analytic space in the natural way. This structure is said to be induced from that of X (cf [Ser55b, 35] for the algebraic case). Similarly, there is a natural analytic structure on the product $X \times X'$ of two analytic spaces, using the product of the charts on X and X'. Given this structure, $X \times X'$ is called the product of the analytic spaces $X \times X'$, and one observes (as above) that the topology is the product of the topologies on X and X'.

We leave the reader to transpose to analytic spaces the other results of [Ser55b, 34-35].

1.3 Analytic sheaves

The definition of analytic sheaves given in [Car, 1951-2 exp. 15] extends to the case of an analytic space X: an analytic sheaf on X is simply a sheaf of $\mathcal{H}_X$-modules.

Let Y be a closed analytic subset of X, and $x \in X$ a point. Denote be $\mathcal{A}_x(Y)$ the set of $f \in \mathcal{H}_{x,X}$ whose restriction to Y vanishes in a neighbourhood of x. The $\mathcal{A}_x(Y)$ form a sheaf of ideals $\mathcal{A}(Y)$ for the sheaf $\mathcal{H}_X$; that is, $\mathcal{A}(Y)$ is an analytic sheaf. The quotient sheaf $\mathcal{H}_X/\mathcal{A}(Y)$ vanishes outside Y, and its restriction to Y is nothing but $\mathcal{H}_Y$, with the usual definition of the induced structure.

Proposition 1. (a) $\mathcal{H}_X$ is a coherent sheaf of rings.

(b) If Y is a closed analytic subset of X, the sheaf $\mathcal{A}(Y)$ is coherent.[3]

In the case that X is an open subset of $\mathbb{C}^n$, these results are due to Oka and Cartan — see [Car50, Theorems 1 and 2] and [Car, 1951-2, exp. 15-16]. The general case procedes immediately; the question is local, so one can assume that X is a closed analytic subset of some open $U \subset \mathbb{C}^n$. In this case, we have $\mathcal{H}_X = \mathcal{H}_U/\mathcal{A}(X)$. In light of the previous, $\mathcal{H}_U$ is a coherent sheaf of rings, and $\mathcal{A}(X)$ is a coherent sheaf of ideals; the result (a) follows by [Ser55b, Theorem 3]. The second assertion is proven in the same way.

Other examples of coherent analytic sheaves include the sheaf of sections to a vector bundle, and the sheaf of automorphic functions [Car, 1953-4, exp. 20].

1.4 Neighbourhood of a point in an analytic space

Let X be an analytic space, x a point of X, and $\mathcal{H}_x$ the ring of germs at x. This ring is a $\mathbb{C}$-algebra which has a unique maximal ideal $\mathfrak{m}$ consisting of those functions vanishing at x, and the field $\mathcal{H}_x/\mathfrak{m}$ is $\mathbb{C}$ — in other words, $\mathcal{H}_x$ is a *local algebra* over $\mathbb{C}$. If $X = \mathbb{C}^n$, then $\mathcal{H}_x = \mathbb{C}\{z_1, \ldots, z_n\}$, the algebra of convergent series in n variables; in the general case, $\mathcal{H}_x$ is isomorphic to a quotient algebra $\mathbb{C}\{z_1, \ldots, z_n\}/\mathfrak{a}$, since

[2] I think this means a category?

[3] In [GLS07], the Oka coherence theorem (a) is Theorem 1.63, and (b) is Theorem 1.75.

X is locally isomorphic to an analytic subset of $\mathbb{C}^n$. As a result, the ring $\mathcal{H}_x$ is noetherian[4], it is also an analytic ring, in the sense of [Car, 1953-4, exp. 8].

We see easily that the knowledge of $\mathcal{H}_x$ determines X in a neighbourhood of x [Car, *loc. cit.*]. In particular, if $\mathcal{H}_x$ is isomorphic to $\mathbb{C}\{z_1, \ldots, z_n\}$ then X is locally isomorphic to $\mathbb{C}^n$; this condition is equivalent to requiring that $\mathcal{H}_x$ is a regular local ring of dimension n[5] (for the theory of local rings, see [Sam53a]). In this case, the point x is called *smooth* of dimension n; if every point of X is smooth, X is called an analytic *variety.*

Returning to the general case, the ring $\mathcal{H}_x$ has no nilpotents other than 0, and as such [Sam53b, Chapter 4 §2]

$$\{0\} = \bigcap \mathfrak{p}_i,$$

where $\mathfrak{p}_i$ runs over the minimal prime ideals of $\mathcal{H}_x$. If we denote by X_i the irreducible components of X containing x, we have $\mathfrak{p}_i = \mathcal{A}_x(X_i)$, and $\mathcal{H})x/\mathfrak{p}_i = \mathcal{H}_{x,X_i}$[6]. This essentially reduces the local study of X to that of X_i; for example, the *dimension* (analytic — that is to say half the topological dimension) of X at x is the largest of the dimensions of the X_i[7]. One observes that this dimension coincides with the dimension (in the Krull sense) of the local ring $\mathcal{H}_x$. To demonstrate this, it suffices to check dimensions in the case that X is irreducible at x (that $\mathcal{H}_x$ is an integral domain) — in this case, if r is the analytic dimension of X at x, we know [Car, 1953-4 exp. 8] that $\mathcal{H}_x$ is a finite extension of $\mathbb{C}\{z_1, \ldots, z_r\}$. Since $\mathbb{C}\{z_1, \ldots, z_r\}$ has completion $\mathbb{C}[\![z_1, \ldots, z_r]\!]$, the same is true of $\mathcal{H}_x$.

2 The analytic space associated to an algebraic variety

In what follows, we consider algebraic varieties over $\mathbb{C}$. Such a variety is given two topologies: the "usual" topology, and the Zariski topology. To avoid confusion, we will prefix notions relative to the latter by the letter Z; for example, "Z-open" is short for "open in the Zariski topology".

2.1 Defenition of the analytic space associated to an algebraic variety

We will give every algebraic variety[8] the structure of an analytic space, which is possible by the following lemma.

Lemma 1. a) The Z-topology on $\mathbb{C}^n$ is less fine that the usual topology.

b) Every Z-locally closed subset of $\mathbb{C}^n$ is analytic.

c) If U and U' are two Z-locally closed subsets of $\mathbb{C}^n$ and $\mathbb{C}^{n'}$, and $f : U \to U'$ is a regular mapping, then f is holomorphic.

d) Under the hypotheses of c), if we suppose in addition that f is a biregular isomorphism, then f is an analytic isomorphism.

By definition, a Z-closed subset of $\mathbb{C}^n$ is defined by the vanishing of a certain number of polynomials; since a polynomial is continuous in the usual topology (resp. holomorphic), one deduces a) (resp. b)). To demonstrate c), we may suppose that $U' = \mathbb{C}$; then we must show that every regular function on U is holomorphic, which follows from the fact that a polynomial is a holomorphic function. Finally, d) follows immediately from c), applied to f^{-1}.

[4] [GLS07, Theorem 1.15].

[5] [GLS07, Proposition 1.48].

[6] See [GLS07, Proposition 1.51], and following exercises

[7] That there is a finite number of irreducible components follows from the fact that $\mathcal{H}_x$ is noetherian — see [GLS07, §B.1].

[8] In "giving X the structure of an analytic space", we mean defining this structure on the underlying set (in a way natural w.r.t. the algebraic structure). It should be noted that to do this for a variety in the scheme sense, one would (I think) need to apply the construction to the set of closed points. See [Har00, Proposition 2.6].

Now let X be an algebraic variety over $\mathbb{C}$ (in the sense of [Ser55b, 34], so not necessarily irreducible). Let V be a Z-open subset of X with an algebraic chart

$$\varphi : V \longrightarrow U,$$

onto a Z-locally closed subset U of some affine space. According to Lemma 1 b), U can be given the structure of an analytic space.

Proposition 2. There exists on X the structure of an analytic space, which is unique if we require that, for every chart $\varphi : V \to U$, the Z-open set V is open, and φ is an analytic isomorphism from V (with the induced analytic structure) onto U (with the analytic structure from Section 1.1).

(More briefly: every algebraic chart is an analytic chart.)

Uniqueness is evident, as we can recover X from the Z-open subsets and their charts. To prove existence, let $\varphi : V \to U$ be a chart, and transport the analytic structure on V to U via φ^{-1}. If $\varphi' : V' \to U'$ is another chart, the analytic structures on $V \cap V'$ induced by V and V' are identical, by Lemma 1 d); in addition, $V \cap V'$ is open in V and in V', by Lemma 1 a). By glueing, we obtain on X a topology and a sheaf $\mathcal{H}_X$ which satisfies the axiom (H1). To verify that our new topology on X is Hausdorff, we use axiom (VA2') of [Ser55b, 34][9]. This axiom implies that the diagonal of X is Z-closed, so *a fortiori* it is closed.

Remark. One can define directly the analytic structure on X without reference to the charts $\varphi : V \to U$. First, one defines the topology to be the finest such that regular functions on Z-open subsets of X are continuous. Then, $\mathcal{H}_{x,X}$ is the analytic subring of $\mathcal{C}(X)_x$ generated by $\mathcal{O}_{x,X}$ (in the sense of [Car, 1953-4, exp. 8]). We leave to the reader to verify the equivalence of the two definitions.

In the following, we denote by X^h the set X with the analytic structure we have defined. The topology on X^h is finer than the topology on X; as X^h can be recovered from a finite number of open subsets with charts[10], X^h is a locally compact space which is *countable at infinity*.

The following properties are immediate from the definition of X^h:

If X and Y are two algebraic varieties, we have $(X \times Y)^h = X^h \times Y^h$. If Y is a Z-locally closed subset of X, then Y^h is an analytic subset of X^h; moreover, the induced analytic structure on Y coincides with the structure on Y^h. Finally, if $f : X \to Y$ is a regular function between algebraic varieties, f is also a holomorphic mapping from X^h to Y^h.

2.2 Relations between the local ring at a point, and the ring of holomorphic functions at that point

Let X be an algebraic variety, and x a point of X. We now compare the local ring $\mathcal{O}_x$ of regular functions at x with the local ring $\mathcal{H}_x$ of germs of holomorphic functions at x[11].

As every regular function is holomorphic, each $f \in \mathcal{O}_x$ defines a germ of a holomorphic function at x, which we denote by $\theta(f)$. The map $\theta : \mathcal{O}_x \to \mathcal{H}_x$ is a homomorphism, and maps the maximal ideal of $\mathcal{O}_x$ into that of $\mathcal{H}_x$. By continuity, it extends to a homomorphism $\hat{\theta} : \hat{\mathcal{O}}_x \to \hat{\mathcal{H}}_x$ between the completions of $\mathcal{O}_x$ and $\mathcal{H}_x$ (see Appendix A.4).

Proposition 3. The homomorphism $\hat{\theta} : \hat{\mathcal{O}}_x \to \hat{\mathcal{H}}_x$ is bijective.

We will demonstrate this proposition concurrently with another result. Let Y be a Z-locally closed subset of X and $\mathcal{J}_x(Y)$ (or $\mathcal{J}_x(Y, X)$ if it is necessary to specify X) the ideal of $\mathcal{O}_x$ formed by functions vanishing on Y in a Z-neighbourhood of x. The image of $\mathcal{J}_x(Y)$ under θ is evidently contained in the ideal $\mathcal{A}_x(Y) \subset \mathcal{H}_x$ defined in Section 1.3.

[9] For charts $\varphi_i : U_i \to V_i$ and $\varphi_j : U_j \to V_j$, define $T_{ij} = \{(\varphi_i(x), \varphi_j(x) \mid x \in U_i \cap U_j\}$. The axiom (VA2') requires that T_{ij} is Z-closed in $V_i \times V_j$, which is equivalent to $\Delta(X) \subset X \times X$ being Z-closed (VA2).

[10] In (VA1) of [Ser55b, 34] the covering of X by algebraic charts is required to be finite.

[11] That is, the stalks of the structure sheaves of X and X^h.

Proposition 4. The ideal $\mathcal{A}_x(Y)$ is generated by $\theta(\mathcal{J}_x(Y))$.

We demonstrate these propositions first in the case where $X = \mathbb{C}^n$. The former is trivial, as $\hat{\mathcal{O}}_x = \hat{\mathcal{H}}_x = \mathbb{C}[\![z_1, \ldots, z_n]\!]$, the formal power series ring in n indeterminants. For Proposition 4, let $\mathfrak{a} \subset \mathcal{H}_x$ be the ideal generated by $\mathcal{J}_x(Y)$. Every ideal of $\mathcal{H}_x$ defines a germ of an analytic subset of X at x (see [Car50, 3] or [Car, 1953-4, exp. 6, p.6]); it is clear that the germ defined by $\mathfrak{a}$ is Y. If f is an element of $\mathcal{A}_x(Y)$, by virtue of the *Nullstellensatz*[12] (which holds for ideals of $\mathcal{H}_x$: [Rü33, p. 278], or [Car, 1951-2, exp. 14, p.3] and [Car, 1953-4, exp. 8, p.9]) there is an integer $r \geq 0$ such that $f^r \in \mathfrak{a}$. *A fortiori*, we have

$$f^r \in \mathfrak{a} \cdot \hat{\mathcal{H}}_x = \mathcal{J}_x(Y) \cdot \hat{\mathcal{H}}_x = \mathcal{J}_x(Y) \cdot \hat{\mathcal{O}}_x.$$

The ideal $\mathcal{J}_x(Y)$ is the intersection of the prime ideals corresponding to the irreducible components of Y at x. By a theorem of Chevalley (see [Sam53a, p.40] and [Sam55, p.67]), the same is true of $\mathcal{J}_x(Y) \cdot \hat{\mathcal{O}}_x$; as such $f^r \in \mathcal{J}_x(Y) \cdot \hat{\mathcal{O}}_x$ implies that $f \in \mathcal{J}_x(Y) \cdot \hat{\mathcal{O}}_x$. As $\mathcal{H}_x$ is a noetherian local ring, we have $\mathfrak{a} \cdot \hat{\mathcal{H}}_x \cap \mathcal{H}_x = \mathfrak{a}$ (see [Sam53b, Chapter 4]). As such, $f \in \mathfrak{a}$ and Proposition 4 for that $X = \mathbb{C}^n$.

Moving to the general case. The question is local, so we may assume that X is a subvariety of some affine space, which we denote by U. By definition, we have

$$\mathcal{O}_x = \mathcal{O}_{x,U}/\mathcal{J}_x(X,U) \qquad \text{and} \qquad \mathcal{H}_x = \mathcal{H}_{x,U}/\mathcal{A}_x(X,U).$$

The mapping $\theta : \mathcal{O}_x \to \mathcal{H}_x$ is induced on the quotient by $\theta : \mathcal{O}_{x,U} \to \mathcal{H}_{x,U}$, and by the previous we know that $\hat{\theta} : \hat{\mathcal{O}}_{x,U} \to \hat{\mathcal{H}}_{x,U}$ is bijective, and that $\mathcal{A}_x(X,U) = \theta(\mathcal{J}_x(X,U)) \cdot \mathcal{H}_{x,U}$. Proposition 3 follows immediately, by Proposition 23. Proposition 4 also follows from the above, since $\mathcal{A}_x(Y)$ is the image of $\mathcal{A}_x(Y,U)$ in the quotient, and the former is generated by $\theta(\mathcal{J}_x(Y,U))$.

Proposition 3 shows, in particular, that $\theta : \mathcal{O}_x \to \mathcal{H}_x$ is injective, so we may identify $\mathcal{O}_x$ with the subring $\theta(\mathcal{O}_x)$ of $\mathcal{H}_x$. Using this identification, we have (using Proposition 22) that:

Corollary 1. The rings $(\mathcal{O}_x, \mathcal{H}_x)$ form a flat pair.

Corollary 2. The rings $\mathcal{O}_x$ and $\mathcal{H}_x$ have the same dimension.

We know that the dimension of a noetherian local ring is equal to that of its completion (see [Sam53a, p.26]).

Taking as given the results of Section 1.4, we obtain the following result (supposing that X is irreducible to simplify the statement):

Corollary 3. If X is an irreducible algebraic variety of dimension r, the analytic space X^h has analytic dimension r at all of its points.

2.3 Relations between the usual and Zariski topologies on an algebraic variety

Proposition 5. Let X be an algebraic variety, and U a subset of X. If U is Z-open and Z-dense in X, U is dense in X.

Let $Y = X \setminus U$, which is a Z-closed subset of X. If there is a neighbourhood of x disjoint from U (ie contained in Y), then $\mathcal{A}_x(Y) = 0$, with the notations of Section 2.2. Since $\theta(\mathcal{J}_x(Y)) \subset \mathcal{A}_x(Y)$, and θ is injective (Proposition 3), we have that $\mathcal{J}_x(Y) = 0$. This indicates that $Y = X$ in a Z-neighbourhood of X, which contradicts the hypothesis that U is Z-dense in X.

Remark. One sees easily that Proposition 5 is equivalent to the fact that $\theta : \mathcal{O}_x \to \mathcal{H}_x$ is injective, a far more elementary fact than Proposition 3. One could demonstrate this fact, for example, by reduction to the case of a curve.

[12] Called the *Hilbert-Rückert Nullstellensatz* in [GLS07, Theorem 1.72].

We now give two simple applications of Proposition 5.

Proposition 6. For an algebraic variety X to be complete, it is necessary and sufficient that it be compact.

We appeal first to a result of Chow (see [Cho56] or [Ser55a, 4]): for every algebraic variety X, there is a projective variety Y, a Z-open and Z-dense subset U of Y, and a surjective regular map $f : U \to X$ whose graph T is Z-closed in $X \times Y$. We have $U = Y$ if and only if X is complete.

Suppose first that X is complete, so that $X = f(Y)$; since every projective variety is compact in the usual topology, we conclude that X is compact. Conversely, if X is compact, the same is true of $T \subset X \times Y$[13]. As U is the projection of T onto Y, U is closed in Y. By Proposition 5, $U = Y$, as required.

The following lemma is essentially due to Chevalley.

Lemma 2. Let $f : X \to Y$ be a regular mapping between algebraic varieties, and suppose that $f(X)$ is Z-dense in Y. Then, there exists $U \subset f(X)$ Z-open and Z-dense in Y.

In the case where X and Y are irreducible, the result is well-known: see [Car, 1955-6, exp. 3] or [Sam55, p.15], for example. We will reduce the general case to this situation. Let X_i, $i \in I$, be the irreducible componenets of X, and Y_i the Z-closure of $f(X_i)$ in Y; the Y_i are irreducible, and $Y = \cup Y_i$. As such, there is a subset $J \subset I$ so that Y_j, $j \in J$ are the irreducible components of Y. By the result mentioned, for each $j \in J$, there is a subset $U_j \subset f(X_j)$ which Z-open and Z-dense in Y_j. Shrinking U_j, we may assume that U_j does not meet Y_k for $j \neq k \in J$. Setting $U = \bigcup_{j \in J} U_j$, we find a subset of Y with the required properties.

Proposition 7. If $f : X \to Y$ is a regular mapping between algebraic varieties, the closure and the Z-closure of $f(X)$ in Y coincide.

Let T be the Z-closure of $f(X)$ in Y. Applying Lemma 2 to $f : X \to T$, we can find $U \subset f(X)$ which is Z-open and Z-dense in T. By Proposition 5, U is dense in T, so *a fortiori* the same is true of $f(X)$. This shows that T contains the closure of $f(X)$, and the opposite inclusion is evident, so the statement is proven.

2.4 An analytic criterion for regularity

We know that every regular function is holomorphic. The following proposition (which we will extend in Section 4.1) indicates when the converse is true.

Proposition 8. Let X and Y be algebraic varieties, and $f : X \to Y$ a holomorphic mapping. If the graph T of f is a Z-locally closed subset (ie an algebraic subvariety) of $X \times Y$, the function f is regular.

Let $p = \mathrm{pr}_X$ be the canonical projection of T onto the first factor X in $X \times Y$. The function p is regular, bijective, and its inverse function $x \mapsto (x, f(x))$ is holomorphic by hypothesis. Therefore, p is an analytic isomorphism, so it suffices to show that p is a biregular isomorphism (so that $f = \mathrm{pr}_Y \circ p^{-1}$). This follows from the following proposition.

Proposition 9. Let T and X be algebraic varieties, and $p : T \to X$ a regular bijective map. If p is an analytic isomorphism of T onto X, it is also a biregular isomorphism.

We show first that p is a homeomorphism for the Zariski topologies on T and X. Let F be a Z-closed subset of T; since p is an analytic isomorphism, it is *a fortiori* a homeomorphism, so $p(F)$ is closed in X. Applying Proposition 7 to $p : F \to X$, we conclude that $p(F)$ is Z-closed in X, which demonstrates our assertion.

[13] As it is closed and $X \times Y$ is compact.

We now show that p transforms the sheaf $\mathcal{O}_X$ to $\mathcal{O}_T$. More precisely, if $t \in T$ is a point, and $x = p(t)$, p defines a homomorphism

$$p^* : \mathcal{O}_{x,X} \longrightarrow \mathcal{O}_{t,T},$$

and we need to show that p^* is bijective[14].

Since p is a Z-homeomorphism, p^* is injective, which permits us to identify $\mathcal{O}_{x,X}$ with a subring of $\mathcal{O}_{t,T}$. To simplify notation, write $A = \mathcal{O}_{x,X}$ and $A' = \mathcal{O}_{t,T}$, so that $A \subset A'$. Similarly, write B (resp. B') for the ring $\mathcal{H}_{x,X}$ (resp. $\mathcal{H}_{t,T}$), and we consider A and A' as embedded in B and B', respectively, by Proposition 3. The hypothesis that p is an analytic isomorphism indicates that $B = B'$.

Let X_i be the irreducible components of X at x; each X_i determines a prime ideal $\mathfrak{p}_i = \mathcal{J}_x(X_i)$ of A, and the (local) quotient ring $A_i = A/\mathfrak{p}_i$ is exactly the local ring at x of X_i. Each field of fractions K_i of A_i is the field of rational functions on the irreducible variety X_i. The ideals $\mathfrak{p}_i$ are evidently the minimal prime ideals of A, and we have that $0 = \bigcap \mathfrak{p}_i$. The set S of elements of A not contained in any $\mathfrak{p}_i$ is mutiplicatively stable (it is easy to see this is the set of regular elements of A). The localisation A_S [15] is equal to the direct product of the K_i (see Lemma 3 following).

Let $T_i = p^{-1}(X_i)$; since p is a Z-homeomorphism, the T_i are the irreducible components of T at t, and define prime ideals $\mathfrak{p}'_i$ of A'. We write again $A'_i = A'/\mathfrak{p}'_i$, and K'_i for the field of fractions of A'_i; again the ring $A'_{S'}$ is the direct product of the K'_i. Note that $\mathfrak{p}'_i \cap A = \mathfrak{p}'_i$, where $A_i \subset A'_i$, $K_i \subset K'_i$ and $A_S \subset A'_{S'}$.

We first show that $K_i = K'_i$, so that p defines a birational correspondence between T_i and X_i. Since $p : T_i \to X_i$ is a Z-homeomorphism, T_i and X_i have the same dimension, so the fields K_i and K'_i have the same transcendence degree over $\mathbb{C}$. If we set $n_i = [K'_i : K_i]$, we know[16] that there exists some non-empty Z-open subset U_i of X_i, so that the inverse image of each point of U_i consists of exactly n_i points of T_i. Since p is bijective $n_i = 1$ and $K_i = K'_i$.

Since A_S (resp. $A'_{S'}$) is the direct product of the K_i (resp. K'_i), we have that $A_S = A'_{S'}$. Now let $f' \in A'$; by the previous, we have $f' \in A_S$ — in other words, there exist $g \in A$ and $s \in S$ so that $g = sf'$. Then $g \in sA'$, so $g \in sB' = sB$. But by Corollary 1, the pair (A, B) is flat, so we have $sB \cap A = sA$ by Proposition 16. This demonstrates that $g \in sA$, so there is $f \in A$ so that $g = sf$, ie $s(f - f') = 0$. Since s is a non-zero-divisor in A, we have $f = f'$ so $A = A'$.

We used the following lemma, which we demonstrate now.

Lemma 3. Let A be a commutative ring, in which the zero ideal is the intersection of a finite number of minimal prime ideals $\mathfrak{p}_i$. Let K_i be the field of fractions of $A/\mathfrak{p}_i$, and S the set of elements not belonging to any $\mathfrak{p}_i$. The ring of fractions A_S is isomorphic to the direct product of the K_i.

We know that the prime ideals of A_S are in bijection with those prime ideals of A which are disjoint from S (see [Sam53b, Chapter 4, §3]). As such, writing $\mathfrak{m}_i = \mathfrak{p}_i A_S$, the $\mathfrak{m}_i$ are the only prime ideals of A_S. In particular, they are minimal, and evidently distinct, since $\mathfrak{m}_i \cap A = \mathfrak{p}_i$ ([Sam53b, *loc. cit.*]). In addition, the field $A_S/\mathfrak{m}_i$ is generated by $A/\mathfrak{p}_i$, so coincides with K_i. It remains to show that the canonical homomorphism

$$\varphi : A_S \longrightarrow \prod A_S/\mathfrak{m}_i = \prod K_i$$

is bijective.

Firstly, since $\bigcap \mathfrak{p}_i = 0$, we have that $\bigcap \mathfrak{m}_i = 0$, which shows that φ is injective. Denote by $\mathfrak{b}_i$ the product (in the ring A_S) of the ideals $\mathfrak{m}_j$, $j \neq i$, and write $\mathfrak{b} = \sum \mathfrak{b}_i$. The ideal $\mathfrak{b}$ is the whole ring, as it is contained in none of the $\mathfrak{m}_i$. Therefore, there exist elements $x_i \in \mathfrak{b}_i$ so that $\sum x_i = 1$. We have:

$$x_i \equiv 1 \mod \mathfrak{m}_i \qquad \text{and} \qquad x_i \equiv 0 \mod \mathfrak{m}_j,\ j \neq i,$$

which shows that $\varphi(A_S)$ contains the elements $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ of $\prod K_i$. Since these elements generate the A_s-module $\prod K_i$, this shows that φ is bijective, which finishes the proof.

[14] Serre credits the following proof to Samuel.

[15] Notation for $S^{-1}A$.

[16] Serre: this is a classical result, and easy to demonstrate, on birational maps. [Sam55, p. 16] has a slightly weaker result, which suffices for our purposes.

3 GAGA theorems

3.1 The analytic sheaf associated to an algebraic sheaf

Let X be an algebraic variety, and X^h the analytic space associated to it in Section 2.1. If $\mathcal{F}$ is a sheaf on X, we give the set $\mathcal{F}$ a new topology making it a sheaf on X^h. This topology is defined in the following way: If $\pi : \mathcal{F} \to X$ denotes the projection of $\mathcal{F}$ onto X, one embeds $\mathcal{F}$ into $X^h \times \mathcal{F}$ by the map $f \mapsto (\pi(f), f)$. The topology on $\mathcal{F}$ in question is that induced from that of $X^h \times \mathcal{F}$. One verifies that this gives the set $\mathcal{F}$ the structure of a sheaf on X^h, which we denote by $\mathcal{F}'$. For every $x \in X$, we have $\mathcal{F}_x = \mathcal{F}'_x$; the sheaves $\mathcal{F}$ and $\mathcal{F}'$ only differ in their topology ($\mathcal{F}'$ is exactly the *inverse image sheaf* of $\mathcal{F}$ under the continuous map $X^h \to X$).

The preceding discussion applies in particular to the sheaf $\mathcal{O}$ of local rings on X; Proposition 3 allows us to identify the sheaf $\mathcal{O}'$ obtained in this way with a subsheaf of $\mathcal{H}$, the sheaf of germs of holomorphic functions on X^h.

Definition 2. Let $\mathcal{F}$ be an algebraic sheaf on X. The sheaf $\mathcal{F}^h$, called the analytic sheaf associated to $\mathcal{F}$, is defined by the formula

$$\mathcal{F}^h = \mathcal{F}' \otimes \mathcal{H},$$

where the tensor product is taken over the sheaf of rings $\mathcal{O}'$.

(In other words, $\mathcal{F}^h$ is obtained from $\mathcal{F}'$ by extension of scalars along $\mathcal{O}' \to \mathcal{H}$.)

The sheaf $\mathcal{F}^h$ is a sheaf of $\mathcal{H}$-modules, that is to say an analytic sheaf. The injection $\mathcal{O}' \to \mathcal{H}$ defines a canonical homomorphism $\alpha : \mathcal{F}' \to \mathcal{F}^h$.

Every algebraic homomorphism (that is to say, $\mathcal{O}$-linear)

$$\varphi : \mathcal{F} \longrightarrow \mathcal{G}$$

defines, by extension of scalars, an analytic homomorphism

$$\varphi^h : \mathcal{F}^h \longrightarrow \mathcal{G}^h.$$

As such, $\mathcal{F}^h$ is a covariant functor of $\mathcal{F}$.

Proposition 10. a) The functor $(-)^h$ is exact.

b) For every algebraic sheaf $\mathcal{F}$, the homomorphism $\alpha : \mathcal{F}' \to \mathcal{F}^h$ is injective.

c) If $\mathcal{F}$ is a coherent algebraic sheaf, $\mathcal{F}^h$ is a coherent analytic sheaf.

If $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is an exact sequence, the same is evidently true of $\mathcal{F}'_1 \to \mathcal{F}'_2 \to \mathcal{F}'_3$. Therefore, the sequence:

$$\mathcal{F}'_1 \otimes \mathcal{H} \to \mathcal{F}'_2 \otimes \mathcal{H} \to \mathcal{F}'_3 \otimes \mathcal{H}$$

is also exact, by Corollary 1, which demonstrates a). The assertion b) follows from the same result.

To demonstrate c), remark first that we have $\mathcal{O}^h = \mathcal{H}$; then, if $\mathcal{F}$ is a coherent algebraic sheaf, and if x is a point of X, we can find an exact sequence:

$$\mathcal{O}^q \longrightarrow \mathcal{O}^p \longrightarrow \mathcal{F} \longrightarrow 0,$$

valid in a Z-neighbourhood U of x. By a), we have an exact sequence

$$\mathcal{H}^q \longrightarrow \mathcal{H}^p \longrightarrow \mathcal{F}^h \longrightarrow 0,$$

also valid on U. Since U is a neighbourhood of $x \in X^h$, and since the sheaf $\mathcal{H}$ is coherent (Proposition 1), this shows that $\mathcal{F}^h$ is coherent [Ser55b, 15].

The previous proposition demonstrates that, in particular, if $\mathcal{J}$ is a sheaf of ideals of $\mathcal{O}$, the sheaf $\mathcal{J}^h$ is exactly the sheaf of ideals of $\mathcal{H}$ generated by the elements of $\mathcal{J}$.

3.2 Extension of a sheaf

Let Y be a Z-closed subvariety of the algebraic variety X, and let $\mathcal{F}$ be a coherent algebraic sheaf on Y. If we denote by $\mathcal{F}^X$ the sheaf obtained by extension by zero on $X \setminus Y$ (see [Ser55b, 5]), we know that $\mathcal{F}^X$ is a coherent algebraic sheaf on X, and the sheaf $(\mathcal{F}^X)^h$ is well-defined; it is a coherent analytic sheaf on X^h. On the other hand, the sheaf $\mathcal{F}^h$ is a coherent analytic sheaf on Y^h, which we can extend by zero on $X^h \setminus Y^h$, obtaining similarly a new sheaf $(\mathcal{F}^h)^X$. We have:

Proposition 11. The sheaves $(\mathcal{F}^h)^X$ and $(\mathcal{F}^X)^h$ are canonically isomorphic.

The two sheaves in questions are zero outside Y^h, so it suffices to show that their restrictions to Y^h are isomorphic.

Let x be a point of Y. Write, to simplify the notation:

$$A = \mathcal{O}_{x,X},\ A' = \mathcal{O}_{x,Y},\ B = \mathcal{H}_{x,X},\ B' = \mathcal{H}_{x,Y},\ E = \mathcal{F}_x.$$

Then we have[17]:

$$(\mathcal{F}^h)^X_x = E \otimes_{A'} B' \quad \text{and} \quad (F^X)^h_x = E \otimes_A B.$$

The ring A' is a quotient of A by an ideal $\mathfrak{a}$, and, by Proposition 4, we have $B' = B/\mathfrak{a}B = B \otimes_A A'$. By the associativity of the tensor product, we obtain an isomorphism:

$$\theta_x : E \otimes_{A'} B' = E \otimes_{A'} A' \otimes_A B \longrightarrow E \otimes_A B,$$

which varies continuously with x, as one sees easily; the proposition follows.

The proposition may be summed up as saying that the functor $\mathcal{F}^h$ is compatible with the usual identification of $\mathcal{F}$ with $\mathcal{F}^X$.

3.3 Induced homomorphisms on cohomology

We use the notations of Section 3.1. Let X be an algebraic variety, $\mathcal{F}$ an algebraic sheaf on X, and $\mathcal{F}^h$ the analytic sheaf associated to $\mathcal{F}$. If U is a Z-open subset of X, and s is a section of $\mathcal{F}$ on U, we can consider s as a section s' of $\mathcal{F}'$ on the open $U^h \subset X^h$. Then, $\alpha(s') = s' \otimes 1$ is a section of $\mathcal{F}^h = \mathcal{F}' \otimes \mathcal{H}$ on U^h. The function $s \mapsto \alpha(s')$ is a homomorphism

$$\epsilon : \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U^h, \mathcal{F}^h).$$

Now let $\mathfrak{U} = \{U_i\}$ be a finite Z-open covering of X; the U_i^h form a finite open covering of X^h, which we denote $\mathfrak{U}^h$. For every collection of indices $i_0, \ldots, i_q$, we have — using the previous — a canonical homomorphism:

$$\epsilon : \Gamma(U_{i_0} \cap \cdots \cap U_{i_1}, \mathcal{F}) \longrightarrow \Gamma(U^h_{i_0} \cap \cdots \cap U^h_{i_1}, \mathcal{F}^h),$$

which gives us a homomorphism

$$\epsilon : C(\mathfrak{U}, \mathcal{F}) \longrightarrow C(\mathfrak{U}^h, \mathcal{F}^h),$$

with the notations of [Ser55b, 18].

This homomorphism commutes with the coboundary d, so defines, by passage to cohomology, new homomorphisms:

$$\epsilon : H^q(\mathfrak{U}, \mathcal{F}) \longrightarrow H^q(\mathfrak{U}^h, \mathcal{F}^h).$$

Finally, by passage to the inductive limit on $\mathfrak{U}$, we obtain the *induced homomorphisms on cohomology groups*

$$\epsilon : H^q(X, \mathcal{F}) \longrightarrow H^q(X^h, \mathcal{F}^h).$$

[17] There is, I think, a typo in the following equation in the original. There it reads $(\mathcal{F}^h)^X_x = E \otimes_A B'$, which doesn't match the subsequent equation.

These homomorphisms enjoy the usual functorial properties; they commute with homomorphisms $\varphi : \mathcal{F} \to \mathcal{G}$; if we have an exact sequence of algebraic sheaves:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0,$$

where the sheaf $\mathcal{A}$ is *coherent*, the diagram:

$$\begin{array}{ccc} H^q(X,\mathcal{C}) & \xrightarrow{\delta} & H^{q+1}(X,\mathcal{A}) \\ \downarrow \epsilon & & \downarrow \epsilon \\ H^q(X,\mathcal{C}) & \xrightarrow{\delta} & H^{q+1}(X,\mathcal{A}) \end{array}$$

is commutative. One can see this, for example, by taking for $\mathfrak{U}$ covers by affine opens (see [Ser55b]).

3.4 Projective Varieties, statements of the Theorems

Suppose that X is a projective variety, that is, a Z-closed sub-variety of some projective space $\mathbb{C}P^r$. Then, we have the following theorems, which we will demonstrate in the following subsections.

Theorem 1. For every coherent algebraic sheaf $\mathcal{F}$ on X, and for every integer $q \geq 0$, the homomorphism

$$\epsilon : H^q(X,\mathcal{F}) \longrightarrow H^q(X^h,\mathcal{F}^h),$$

defined in Section 3.3, is bijective.

In particular, for $q = 0$ we obtain an isomorphism of $\Gamma(X,\mathcal{F})$ onto $\Gamma(X^h,\mathcal{F}^h)$.

Theorem 2. If $\mathcal{F}$ and $\mathcal{G}$ are two coherent algebraic sheaves on X, every analytic homomorphism $\mathcal{F}^h \to \mathcal{G}^h$ arises from a unique algebraic homomorphism $\mathcal{F} \to \mathcal{G}$.

Theorem 3. For every coherent analytic sheave $\mathcal{M}$ on X^h, there is a coherent algebraic sheaf $\mathcal{F}$ on X such that $\mathcal{F}^h$ is isomorphic to $\mathcal{M}$. In addition, this property determines $\mathcal{F}$ uniquely up to isomorphism.

Remark. 1. These three theorems signify that the theory of coherent analytic sheaves on X^h coincides essentially with the theory of coherent algebraic sheaves on X. Note that they are given for a *projective* variety X, but are exactly the same for an affine variety.

2. We can factorise ϵ as:

$$H^q(X,\mathcal{F}) \longrightarrow H^q(X^h,\mathcal{F}') \longrightarrow H^q(X^h,\mathcal{F}^h).$$

One might ask where $H^q(X,\mathcal{F}) \to H^q(X^h,\mathcal{F}')$ is bijective. The response is negative. If this homomorphism were bijective for every coherent algebraic sheaf $\mathcal{F}$, it would also be so for the constant sheaf $K = C(X)$ of rational functions on X (supposed to be irreducible), since this sheaf is a union of coherent sheaves (compare with [Ser55a, §2]). In this case, $H^q(X,K) = 0$ for every $q > 0$, but $H^q(X^h,K)$ is a K-vector space with dimension equal to the q^{th} Betti number of X^h.

3.5 Proof of Theorem 1

Let us suppose that X is embedded in $\mathbb{C}P^r$; if we identify $\mathcal{F}$ with the sheaf obtained by extending by zero outside X, we have [Ser55b, 26] that:

$$H^q(X,\mathcal{F}) = H^q(\mathbb{C}P^r,\mathcal{F}) \quad \text{and} \quad H^q(X^h,\mathcal{F}^h) = H^q((\mathbb{C}P^r)^h,\mathcal{F}^h),$$

where the notation $\mathcal{F}^h$ is justified by Proposition 11. One sees that it suffices to prove that

$$\epsilon : H^q(\mathbb{C}P^r,\mathcal{F}) \longrightarrow H^q((\mathbb{C}P^r)^h,\mathfrak{F}^h),$$

is bijective. In other words, we reduce to the case that $X = \mathbb{C}P^r$.

First we establish two lemmas.

Lemma 4. Theorem 1 is true for the sheaf $\mathcal{O}$.

If $q = 0$, $H^0(X, \mathcal{O})$ and $H^0(X^h, \mathcal{O}^h)$ are both reduced to constants. If $q > 0$, we know that $H^q(X, \mathcal{O}) = 0$ by [Ser55b, 65, Proposition 8]. On the other hand, by a theorem of Dolbeaut [Dol53], $H^q(X^h, \mathcal{O}^h)$ is isomorphic to the $(0, q)$-type cohomology of the projective space X, which also vanishes.[18]

Lemma 5. Theorem 1 is true for the sheaves $\mathcal{O}(n)$.

For the definition of $\mathcal{O}(n)$, see [Ser55b, 16, or 54].

We reason by induction on $r = \dim X$, the case where $r = 0$ being trivial. Let t be a linear form which is not identically zero, defined by homogenous coordinates $t_0, \ldots, t_r$, and let E be the hyperplance defined by the equation $t = 0$. We have an exact sequence:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

where $\mathcal{O} \to \mathcal{O}_E$ is given by restriction, and $\mathcal{O}(-1) \to \mathcal{O}$ by multiplication by t (see [Ser55b, 81]). From this, we deduce an exact sequence, valid for all $n \in \mathbb{Z}$:

$$0 \longrightarrow \mathcal{O}(n-1) \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}_E(n) \longrightarrow 0.$$

By Section 3.3, we have a commutative diagram:

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & H^q(X, \mathcal{O}(n-1)) & \longrightarrow & H^q(X, \mathcal{O}(n)) & \longrightarrow & H^q(X, \mathcal{O}_E(n-1)) & \longrightarrow & H^{q+1}(X, \mathcal{O}(n-1)) & \longrightarrow \ldots \\
& \downarrow\epsilon & & \downarrow\epsilon & & \downarrow\epsilon & & \downarrow\epsilon & \\
\ldots \longrightarrow & H^q(X^h, \mathcal{O}^h(n-1)) & \longrightarrow & H^q(X^h, \mathcal{O}^h(n)) & \longrightarrow & H^q(X^h, \mathcal{O}_E^h(n-1)) & \longrightarrow & H^{q+1}(X^h, \mathcal{O}^h(n-1)) & \longrightarrow \ldots
\end{array}$$

By the inductive hypothesis, the homomorphism

$$\epsilon : H^q(E, \mathcal{O}_E) \longrightarrow H^q(E^h, \mathcal{O}_E(n)^h)$$

is bijective for all $q \geq 0$ and $n \in \mathbb{Z}$. Applying the Five lemma, we see that if Theorem 1 is true for $\mathcal{O}(n)$, it is true for $\mathcal{O}(n-1)$, and vice-versa. Since it is true for $n = 0$ by Lemma 4, it is true for every n.

We now proceed to the proof of Theorem 1: we reason by descending induction on q. Since $H^q(X, \mathcal{F}) = H^q(X^h, \mathcal{F}^h) = 0$ for $q > 2r$, the theorem is trivial in that case. By [Ser55b, 55, Corollary to Theorem 1], there exists a short exact sequence of coherent algebraic sheaves:

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{L} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{L}$ is a direct sum of sheaves isomorphic to $\mathcal{O}(n)$. By Lemma 5, Theorem 1 is true for the sheaf $\mathcal{L}$.

We have a commutative diagram:

$$\begin{array}{ccccccccc}
H^q(X, \mathcal{R}) & \longrightarrow & H^q(X, \mathcal{L}) & \longrightarrow & H^q(X, \mathcal{F}) & \longrightarrow & H^{q+1}(X, \mathcal{R}) & \longrightarrow & H^{q+1}(X, \mathcal{L}) \\
\downarrow\epsilon_1 & & \downarrow\epsilon_2 & & \downarrow\epsilon_3 & & \downarrow\epsilon_4 & & \downarrow\epsilon_5 \\
H^q(X^h, \mathcal{R}^h) & \longrightarrow & H^q(X^h, \mathcal{L}^h) & \longrightarrow & H^q(X^h, \mathcal{F}^h) & \longrightarrow & H^{q+1}(X^h, \mathcal{R}^h) & \longrightarrow & H^{q+1}(X^h, \mathcal{L}^h)
\end{array}$$

In this diagram, the homomorphisms ϵ_4 and ϵ_5 are bijective, by the inductive hypothesis. By the previous, the Five lemma implies that ϵ_3 is surjective. This result is true for any coherent algebraic sheaf $\mathcal{F}$, so in particular to $\mathcal{R}$, which shows that ϵ_1 is surjective. Another application of the Five lemma shows that ϵ_3 is bijective, which finishes the proof.

[18] Serre: One can calculate $H^q(X, \mathcal{O})$ directly using the open cover of X defined in Section 3.8 and a Laurent series development (J. Frenkel, unpublished). In this way one avoids any recourse to the theory of Kähler manifolds.

3.6 Proof of Theorem 2

Let $\mathcal{A} = \operatorname{Hom}(\mathcal{F}, \mathcal{G})$, the sheaf of germs of homomorphisms $\mathcal{F} \to \mathcal{G}$ (see [Ser55b, 11,14]). An element $f \in \mathcal{A}_x$ is a germ of some homomorphism $\mathcal{F} \to \mathcal{G}$, in a neighbourhood of x, so defines a germ of a homomorphism f^h from the analytic sheaf $\mathcal{F}^h$ to $\mathcal{G}^h$. The map $f \mapsto f^h$ is an $\mathcal{O}'$-linear homomorphism from the sheaf $\mathcal{A}'$ defined by $\mathcal{A}$ (see Section 3.1) to the sheaf $\mathcal{B} = \operatorname{Hom}(\mathcal{F}^h, \mathcal{G}^h)$. This homomorphism extends by linearity to a homomorphism

$$\iota : \mathcal{A}^h \longrightarrow \mathcal{B}.$$

Lemma 6. The homomorphism $\iota : \mathcal{A}^h \to \mathcal{B}$ is bijective.

Let $x \in X$. Since $\mathcal{F}$ is coherent, we have by [Ser55b, 14]:

$$\mathcal{A}_x = \operatorname{Hom}(\mathcal{F}_x, \mathcal{G}_x) \quad \text{and so} \quad \mathcal{A}^h_x = \operatorname{Hom}(\mathcal{F}_x, \mathcal{G}_x) \otimes \mathcal{H}_x;$$

the functors $\otimes$ and Hom are over the ring $\mathcal{O}_x$.

Since $\mathcal{F}^h$ is coherent, where have, in the same way:

$$\mathcal{B}_x = \operatorname{Hom}(\mathcal{F}_x \otimes \mathcal{H}_x, \mathcal{G}_x \otimes \mathcal{H}_x),$$

where the functor $\otimes$ is over $\mathcal{O}_x$, and the functor Hom over $\mathcal{H}_x$.

This implies that the homomorphism:

$$\iota_x : \operatorname{Hom}(\mathcal{F}_x, \mathcal{G}_x) \otimes \mathcal{H}_x \longrightarrow \operatorname{Hom}(\mathcal{F}_x \otimes \mathcal{H}_x, \mathcal{G}_x \otimes \mathcal{H}_x)$$

is bijective, since the pair $(\mathcal{O}_x, \mathcal{H}_x)$ is flat and using Proposition 15.

We now demonstrate Theorem 2. Consider the homomorphisms:

$$H^0(X, \mathcal{A}) \xrightarrow{\epsilon} H^0(X^h, \mathcal{A}^h) \xrightarrow{\iota} H^0(X^h, \mathcal{B}).$$

An element of $H^0(X, \mathcal{A})$ (resp. of $H^0(X^h, \mathcal{B})$) is a homomorphism $\mathcal{F} \to \mathcal{G}$ (resp. $\mathcal{F}^h \to \mathcal{G}^h$). Moreover, if $f \in H^0(X, \mathcal{A})$ we have $\iota \circ \epsilon(f) = f^h$, by the definition of ι. Theorem 2 reduces to showing that $\iota \circ \epsilon$ is bijective. The map ϵ is bijective by Theorem 1 (which applies since $\mathcal{A}$ is coherent, by [Ser55b, 14]), and ι is bijective by Lemma 6.

3.7 Proof of Theorem 3: preliminaries

The *uniqueness* of the sheaf $\mathcal{F}$ follows from Theorem 2. If $\mathcal{F}$ and $\mathcal{G}$ are two coherent algebraic sheaves fulfilling the statement, there exists by hypothesis an isomorphism $g : \mathcal{F}^h \to \mathcal{G}^h$. By Theorem 2, there is a homomorphism $f : \mathcal{F} \to \mathcal{G}$ so that $g = f^h$. If we denote by $\mathcal{A}$ and $\mathcal{B}$ the kernel and cokernel of f, we have an exact sequence:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \longrightarrow \mathcal{B} \longrightarrow 0,$$

from which we obtain, by Proposition 10 a), an exact sequence:

$$0 \longrightarrow \mathcal{A}^h \longrightarrow \mathcal{F}^h \xrightarrow{g} \mathcal{G}^h \longrightarrow \mathcal{B}^h \longrightarrow 0.$$

Since g is bijective, this implies that $\mathcal{A}^h = \mathcal{B}^h = 0$, which by Proposition 10 b) implies that $\mathcal{A} = \mathcal{B} = 0$, showing that f is bijective.

It remains to demonstrate the existence of $\mathcal{F}$. I claim that we may restrict ourselves to the case where X is a projective space $\mathbb{C}P^r$. To show this, let Y be an algebraic subvariety of $X = \mathbb{C}P^r$, and $\mathcal{M}$ a coherent analytic sheaf of Y^h. The sheaf $\mathcal{M}^X$ obtained by extending $\mathcal{M}$ by 0 outside Y^h is a coherent analytic sheaf on X^h. If we take as given Theorem 3 for the space X, there exists a coherent algebraic sheaf $\mathcal{G}$ on X such that $\mathcal{G}^h$ is isomorphic to $\mathcal{M}^X$. Let $\mathcal{J} = \mathcal{J}(Y)$, the coherent sheaf of ideals defined by the subvariety Y. If $f \in \mathcal{J}_x$, multiplication by f is an endomorphism φ of $\mathcal{G}_x$; the endomorphism φ^h of

$\mathcal{G}^h_x = \mathcal{M}^X_x$ is zero, since $\mathcal{M}$ is a coherent analytic sheaf on Y^h. By Proposition 10 b), the same is true of φ. Therefore, we have that $\mathcal{J} \cdot \mathcal{G} = 0$, which indicates there exists a coherent algebraic sheaf $\mathcal{F}$ on Y, such that $\mathcal{G} = \mathcal{F}^X$ [Ser55b, 39, Proposition 3]. By Proposition 11, $(\mathcal{F}^h)^X$ is isomorphic to $(\mathcal{F}^X)^h = \mathcal{G}^h$, which is isomorphic to $\mathcal{M}^X$. By restriction to Y, we see that $\mathcal{F}^h$ is isomorphic to $\mathcal{M}$, which demonstrates our assertion.

3.8 Proof of Theorem 3: the sheaves $\mathcal{M}(n)$

By the previous subsection, we may suppose that $X = \mathbb{C}P^r$, and reason by induction on r, the case where $r = 0$ being trivial.

For every $n \in \mathbb{Z}$, we first define a new analytic sheaf $\mathcal{M}(n)$.

Let $t_0, \ldots, t_r$ be a system of homogenous coordinates on X, and let U_i be the open set where $t_i \neq 0$. We denote by $\mathcal{M}_i$ the restriction of the sheaf $\mathcal{M}$ to U_i. Multiplication by t^n_j/t^n_i is an isomorphism of $\mathcal{M}_j$ onto $\mathcal{M}_i$, defined on $U_i \cap U_j$. The sheaf $\mathcal{M}(n)$ is defined by glueing the sheaves $\mathcal{M}_i$ along these isomorphisms (compare [Ser55b, 54], where the same construction is applied to algebraic sheaves). The sheaf $\mathcal{M}(n)$ is locally isomorphic to $\mathcal{M}$, so is coherent as $\mathcal{M}$ is; we have a canonical isomorphism $\mathcal{M}(n) = \mathcal{M} \otimes \mathcal{H}(n)$, where the tensor product is taken over $\mathcal{H}$. If $\mathcal{F}$ is an algebraic sheaf, we have that $\mathcal{F}^h(n) = \mathcal{F}(n)^h$.

Lemma 7. Let E be a hyperplane in $\mathbb{C}P^r$, and let $\mathcal{A}$ be a coherent analytic sheaf on E. We have $H^q(E^h, \mathcal{A}(n)) = 0$ for $q > 0$ and n sufficiently large.

(This is "Theorem B" of [Car, 1953-4, exp. 18].)

By the inductive hypothesis, there exists a coherent algebraic sheaf $\mathcal{F}$ on E such that $\mathcal{A} = \mathcal{F}^h$, and we have $\mathcal{A}(n) = \mathcal{F}(n)^h$. By Theorem 1, $H^q(E^h, \mathcal{A}(n))$ is isomorphic to $H^q(E, \mathcal{F}(n))$, and the lemma follows from [Dol53, 65, Proposition 7].

Lemma 8. Let $\mathcal{M}$ be a coherent analytic sheaf of $X = \mathbb{C}P^r$. There exists an integer $n(\mathcal{M})$ such that, for every $n \geq (\mathcal{M})$, and for every $x \in X$, the $\mathcal{H}_x$-modules $\mathcal{M}(n)_x$ is generated by elements of $H^0(X^h, \mathcal{M}(n))$.

(This is "Theorem A" of [Car, 1953-4, exp. 18].)

First we remark that, if $H^0(X^h, \mathcal{M}(n))$ is generated by $\mathcal{M}(n)_x$, the same property is true for every $m \geq n$. To this effect, let k be an index such that $x \in U_k$; for every i, let θ_i be the homothety of ratio $(t_k/t_i)^{m-n}$ on $\mathcal{M}_i$. The θ_i commute with the identifications which define $\mathcal{M}(n)$ and $\mathcal{M}(m)$ respectively, so give rise to a homomorphism $\theta : \mathcal{M}(n) \to \mathcal{M}(m)$. Since θ is an isomorphism on U_k, our assertion follows.

Remark also that, is $H^0(X^h, \mathcal{M}(n))$ generates $\mathcal{M}(n)_x$, it also generates $\mathcal{M}(n)_y$ for y sufficiently close to x, by [Ser55b, 12].

These two remarks reduce us to demonstrate the following statement: *for every $x \in X$, there is an integer n, dependent on x and on $\mathcal{M}$, such that $H^0(X^h, \mathcal{M}(n))$ generates $\mathcal{M}(n)_x$.*

Choose a hyperplane E passing through x, with homogenous equation $t = 0$. If $\mathcal{A}(E)$ denotes the sheaf of ideals defined by E (see Section 1.3), we have an exact sequence:

$$0 \longrightarrow \mathcal{A}(E) \longrightarrow \mathcal{H} \longrightarrow \mathcal{H}_E \longrightarrow 0.$$

In addition, the sheaf $\mathcal{A}(E)$ is isomorphic to $\mathcal{H}(-1)$, where the isomorphism $\mathcal{H}(-1) \to \mathcal{A}(E)$ is defined by multiplication by t (see the proof of Lemma 5).

By tensoring with $\mathcal{M}$, we find an exact sequence:

$$\mathcal{M} \otimes \mathcal{A}(E) \longrightarrow \mathcal{M} \longrightarrow \mathcal{M} \otimes \mathcal{H}_E \longrightarrow 0.$$

Denote by $\mathcal{B}$ the sheaf $\mathcal{M} \otimes \mathcal{H}_E$, and let $\mathcal{C}$ be the kernel of the homomorphism $\mathcal{M} \otimes \mathcal{A}(E) \to \mathcal{M}$ (we have $\mathcal{C} = \mathrm{Tor}_1(\mathcal{M}, \mathcal{H}_E)$). Since $\mathcal{A}(E)$ is isomorphic to $\mathcal{H}(-1)$, the sheaf $\mathcal{M} \otimes \mathcal{A}(E)$ is isomorphic to $\mathcal{M}(-1)$, and we obtain an exact sequence:

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{M}(-1) \longrightarrow \mathcal{M} \longrightarrow \mathcal{B} \longrightarrow 0. \qquad (1)$$

Applying the functor $\mathcal{M}(n)$ to the exact sequence eq. (1), we obtain a new exact sequence:

$$0 \longrightarrow \mathcal{C}(n) \longrightarrow \mathcal{M}(n-1) \longrightarrow \mathcal{M}(n) \longrightarrow \mathcal{B}(n) \longrightarrow 0. \tag{2}$$

Let $\mathcal{L}_n$ be the kernel of the homomorphism $\mathcal{M}(n) \to \mathcal{B}(n)$; the sequence eq. (2) decomposes into two exact sequences:

$$0 \longrightarrow \mathcal{C}(n) \longrightarrow \mathcal{M}(n-1) \longrightarrow \mathcal{L}_n \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow \mathcal{L}_n \longrightarrow \mathcal{M}(n) \longrightarrow \mathcal{B}(n) \longrightarrow 0, \tag{4}$$

which, in their turn, give rise to exact sequences in cohomology:

$$H^1(X^h, \mathcal{M}(n-1)) \longrightarrow H^1(X^h, \mathcal{L}_n) \longrightarrow H^2(X^h, \mathcal{C}(n)) \tag{5}$$

and

$$H^1(X^h, \mathcal{L}_n) \longrightarrow H^1(X^h, \mathcal{M}(n) \longrightarrow H^1(X^h, \mathcal{B}(n)). \tag{6}$$

By the definitions of $\mathcal{B}$ and $\mathcal{C}$, we have $\mathcal{A}(E) \cdot \mathcal{B} = 0$ and $\mathcal{A}(E) \cdot \mathcal{C} = 0$, which indicates that $\mathcal{B}$ and $\mathcal{C}$ are coherent analytic sheaves on the hyperplane E. Applying Lemma 7, we see that for some integer n_0 and every $n \geq n_0$, $H^1(X^h, \mathcal{B}(n)) = 0$ and $H^2(X^h, \mathcal{C}(n)) = 0$. Then, the exact sequences eqs. (5) and (6) give us the inequalities:

$$\dim H^1(X^h, \mathcal{M}(n-1)) \geq \dim H^1(X^h, \mathcal{L}_n) \geq \dim H^1(X^h, \mathcal{M}(n)). \tag{7}$$

These dimesions are *finite*, by [CS53] (see also [Car, 1953-4, exp. 17]). As a result, $\dim H^1(X^h, \mathcal{M}(n))$ is a *decreasing* function on n, for $n \geq n_0$; as such, there exists an integer $n_1 \geq n_0$ such that the function $\dim H^1(X^h, \mathcal{M}(n))$ is *constant* for $n \geq n_1$. Then we have[19]:

$$\dim H^1(X^h, \mathcal{M}(n-1)) = \dim H^1(X^h, \mathcal{L}_n) = \dim H^1(X^h, \mathcal{M}(n)) \quad \text{for } n > n_1. \tag{8}$$

Since $n_1 \geq n_0$, we have $H^1(X^h, \mathcal{B}(n)) = 0$, and the exact sequence eq. (6) shows that $H^1(X^h, \mathcal{L}_n) \to H^1(X^h, \mathcal{M}(n))$ is surjective. However, by eq. (8), these two vector spaces have the same dimension; the homomorphism in question is then injective, and the exact sequence eq. (4) shows that[20]

$$H^0(X^h, \mathcal{M}(n)) \longrightarrow H^0(X^h, \mathcal{B}(n)) \quad \text{is surjective for } n > n_1. \tag{9}$$

Now we choose an integer $n > n_1$ so that $H^0(X^h, \mathcal{B}(n))$ generates $\mathcal{B}(n)_x$; this is possible, as $\mathcal{B}$ is a coherent analytic sheaf on E, so has the form $\mathcal{G}^h$, which implies that $H^0(X^h, \mathcal{B}(n)) = H^0(X, \mathcal{G}(n))$ by Theorem 1, and we know that $H^0(X, \mathcal{G}(n))$ generates $\mathcal{G}(n)_x$ for n sufficiently large, by [Ser55b, 55, Theorem 1].

Having done this, I claim that this integer n satisfies our requirements. To demonstrate this, write, to simplify notation, $A = \mathcal{H}_x$, $M = \mathcal{M}(n)_x$, $\mathfrak{p} = \mathcal{A}_x(E)$, and let N be the sub-A-module of M generated by $H^0(X^h, \mathcal{M}(n))$. We have

$$\mathcal{B}(n)_x = \mathcal{M}(n)_x \otimes \mathcal{H}_{x,E} = M \otimes_A A/\mathfrak{p} = M/\mathfrak{p}M.$$

On the other hand, the preceding discussion implies that the canonical image of N in $M/\mathfrak{p}M$ generates $M/\mathfrak{p}M$. This may be written $M = N + \mathfrak{p}M$, which implies, *a fortiori*, $M = N + \mathfrak{m}M$ ($\mathfrak{m}$ denoting the maximal ideal of the local ring A). This implies that $M = N$ (Corollary 5). This finishes the proof of Lemma 8.

[19] There is a small typo here in the original, changing the first $n-1$ to n.

[20] Serre: one recognises the procedure used by Kodaira-Spencer for demonstrating Lefschetz' Theorem [KS53].

3.9 Completing the proof of Theorem 3

Let, as before, $\mathcal{M}$ be a coherent analytic sheaf on $X = \mathbb{C}P^r$. In light of Lemma 8, there is an integer n such that $\mathcal{M}(n)$ is isomorphic to a quotient of some sheaf $\mathcal{H}^p$, and as such $\mathcal{M}$ is isomorphic to a quotient on $\mathcal{H}(-n)^p$. If we denote by $\mathcal{L}_0$ the coherent algebraic sheaf $\mathcal{O}(-n)^p$, we find an exact sequence:

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{L}_0^h \longrightarrow \mathcal{M} \longrightarrow 0,$$

where $\mathcal{R}$ is a coherent analytic sheaf.

Applying the same reasoning to $\mathcal{R}$, we construct a coherent algebraic sheaf $\mathcal{L}_1$, and a surjective analytic homomorphism $\mathcal{L}_1^h \to \mathcal{R}$. From this, we obtain an exact sequence:

$$\mathcal{L}_1^h \xrightarrow{g} \mathcal{L}_0^h \longrightarrow \mathcal{M} \longrightarrow 0.$$

By Theorem 2, there is a homomorphism $f : \mathcal{L}_1 \to \mathcal{L}_2$ such that $g = f^h$. If we denote by $\mathcal{F}$ the cokernel of f, we have an exact sequence:

$$\mathcal{L}_1 \xrightarrow{f} \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

from which (Proposition 10), a final exact sequence:

$$\mathcal{L}_1^h \xrightarrow{g} \mathcal{L}_0^h \longrightarrow \mathcal{F}^h \longrightarrow 0,$$

which demonstrates that $\mathcal{M}$ is isomorphic to $\mathcal{F}^h$, finishing the proof.

4 Applications

4.1 Chow's theorem

Proposition 12 (Chow's theorem)**.** Every closed analytic subset of projective space is algebraic.

We show how this result follows from Theorem 3. Let X be some projective space, and Y a closed analytic subset of X^h. By a theorem of Cartan (Proposition 1), the sheaf $\mathcal{H}_Y = \mathcal{H}_X/\mathcal{A}(Y)$ is a coherent sheaf on X^h; by Theorem 3 there exists a coherent algebraic sheaf $\mathcal{F}$ on X so that $\mathcal{F}^h = \mathcal{H}_Y$. By Proposition 10 b), the support of $\mathcal{F}^h$ is equal to that of $\mathcal{F}$ (recall that, as in [Ser55b, 81], the support of $\mathcal{F}$ is the set of $x \in X$ such that $\mathcal{F}_x \neq 0$), so is Z-closed as $\mathcal{F}$ is coherent. Since $\mathcal{F}^h = \mathcal{H}_Y$, this implies that Y is Z-closed.

We indicate now some simple applications of Chow's theorem.

Proposition 13. If X is an algebraic variety, every compact analytic subset X' of X is algebraic.

Recall the notations of the proof of Proposition 6: let Y be a projective variety, U a Z-open and Z-dense subset of Y, and $f : U \to X$ a surjective regular map such whose graph T is Z-closed in $X \times Y$. Let $T' = T \cap (X' \times Y)$; since X' and Y are compact, and T is closed, T' is compact; as such, the same is true of the projection Y' of T' onto the factor Y. On the other hand, $Y' = f^{-1}(X')$, which implies that Y' is a closed analytic subset of U, and as such of Y. Chow's theorem implies that Y' is a Z-closed subset of Y. Applying Proposition 7 to $f : Y' \to X$, we conclude that $X' = f(Y')$ is Z-closed in X.

Proposition 14. Every holomorphic function f from a compact algebraic variety X to a variety Y is regular.

Let T be the graph of f in $X \times Y$. Since f is holomorphic, T is a compact analytic subset of $X \times Y$; Proposition 13 implies that T is algebraic, which implies that f is regular, by Proposition 8.

Corollary 4. Every compact analytic space has, in addition, the structure of an algebraic variety.

A Results on local rings

All rings considered below will be commutative and unital; all modules over these rings will be unitary.

A.1 Flat modules

Definition 3. Let B be an A module. We say that B is A-flat (or flat) if, for every exact sequence of A-modules:

$$E \longrightarrow F \longrightarrow G,$$

the sequence

$$E \otimes_A B \longrightarrow F \otimes_A B \longrightarrow G \otimes_A B$$

is exact.

Examining the definition of the functors Tor, the preceding condition is equivalent to saying that $\mathrm{Tor}_1^A(B, Q) = 0$ for every A-module Q; since Tor commutes with inductive limits, we can restrict to the case of finite-type modules Q, and similarly (by the exact sequence for Tor) to modules generated by a single element. Therefore, for B to be A-flat, it is necessary and sufficient that $\mathrm{Tor}_1^A(B, A/\mathfrak{a}) = 0$ for every ideal $\mathfrak{a}$ of A — in other words that the canonical homomorphism $\mathfrak{a} \otimes_A B \to B$ is injective.

Example 1. 1. If A is a principal ideal domain, the previous implies that "B is A-flat" is equivalent to "B is torsion-free".

2. If S is a multiplicatively-closed subset of A, the ring of fractions A_S is A-flat, by [Ser55b, 48, Lemma 1].

Let A and B be two rings, and let $\theta : A \to B$ be a homomorphism; this homomorphism gives B the structure of an A-module. If E and F are two A-modules, $E \otimes_A B$ and $F \otimes_A B$ have the structure of B-modules; in addition, if $f : E \to F$ is a homomorphism, $f \otimes 1$ is a B-homomorphism from $E \otimes_A B$ to $F \otimes_A B$. Therefore, we obtain a canonical A-linear map:

$$\mathrm{Hom}_A(E, F) \longrightarrow \mathrm{Hom}_B(E \otimes_A B, F \otimes_A B),$$

which extends by linearity to a B-linear map:

$$\iota : \mathrm{Hom}_A(E, F) \otimes_A B \longrightarrow \mathrm{Hom}_B(E \otimes_A B, F \otimes_A B).$$

Proposition 15. The homomorphism ι defined above is bijective in the case that A is a noetherian ring, E is a finite-type A-module, and B is A-flat.

For a fixed module F, write:

$$T(E) = \mathrm{Hom}_A(E, F) \otimes_A B \quad \text{and} \quad T'(E) = \mathrm{Hom}_B(E \otimes_A B, F \otimes_A B),$$

so that ι is a morphism of functors from $T(E)$ to $T'(E)$.

For $E = A$, we have $T(E) = T'(E) = F \otimes_A B$, and ι is bijective; the same is true if E is a free module of finite type.

Since A is noetherian, and E has finite type, there is an exact sequence:

$$L_1 \longrightarrow L_0 \longrightarrow E \longrightarrow 0,$$

where L_0 and L_1 are free modules of finite type. Consider the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(E) & \longrightarrow & T(L_0) & \longrightarrow & T(L_1) \\ & & \downarrow \iota & & \downarrow \iota_0 & & \downarrow \iota_1 \\ 0 & \longrightarrow & T'(E) & \longrightarrow & T'(L_0) & \longrightarrow & T'(L_1). \end{array}$$

The first line is exact since B is A-flat, and the second is also by general properties of the functors $\otimes$ and Hom. Since we know that ι_0 and ι_1 are bijective, it follows that ι is bijective.

A.2 Flat pairs

Definition 4. A pair of rings $A \subset B$ is called *flat* if B/A is a flat A-module.

Proposition 16. For a pair (A, B) of rings to be flat, it is necessary and sufficient that B is A-flat, and that one of the following properties are satisfied:

- a) (resp. a')) For every A-module (resp. every finite-type A-module) E, the homomorphism $E \to E \otimes_A B$ is injective.
- a") For every ideal $\mathfrak{a}$ of A, $\mathfrak{a}B \cap A = \mathfrak{a}$.

If E is an A-module, the exact sequence:

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0,$$

gives rise to an exact sequence:

$$\mathrm{Tor}_1^A(A, E) \longrightarrow \mathrm{Tor}_1^A(B, E) \longrightarrow \mathrm{Tor}_1^A(B/A, E) \longrightarrow A \otimes_A E \longrightarrow B \otimes_A E.$$

Since $A \otimes_A E = E$ and $\mathrm{Tor}(A, E) = 0$, we obtain a new exact sequence:

$$0 \longrightarrow \mathrm{Tor}_1^A(B, E) \longrightarrow \mathrm{Tor}_1^A(B/A, E) \longrightarrow E \longrightarrow B \otimes_A E.$$

From this, we observe that, for $\mathrm{Tor}_1^A(B/A, E)$ to be zero, it is necessary and sufficient that the same is true of $\mathrm{Tor}_1^A(B, E)$ and that $E \to E \otimes_A B$ is injective. The proposition follows immediately (note that the property a") reduces to the statement that $A/\mathfrak{a} \to A/\mathfrak{a} \otimes_A B$ is injective).

Proposition 17. Let $A \subset B \subset C$ be three rings. If the pairs (A, C) and (B, C) are flat, the same is true of (A, B).

We first show that B is A-flat, in other words, that given an exact sequence of A-modules:

$$0 \longrightarrow E \longrightarrow F,$$

the sequence $0 \to E \otimes_A B \to F \otimes_A B$ is also exact.

Let N be the kernel of the homomorphism $E \otimes_A B \to F \otimes_A B$; since C is B-flat, we have an exact sequence:

$$0 \longrightarrow N \otimes_B C \longrightarrow (E \otimes_A B) \otimes_B C \longrightarrow (F \otimes_A B) \otimes_B C.$$

However, by the associativity of the tensor product, $(E \otimes_A B) \otimes_B C$ is identified with $E \otimes_A C$, and $(F \otimes_A B) \otimes_B C$ with $F \otimes_A C$. In addition, C being A-flat, the homomorphism $E \otimes_A C \to F \otimes_A C$ is injective. It follows that $N \otimes_B C = 0$, and, applying Proposition 16 to the pair (B, C), we see that $N = 0$, which shows that B is A-flat.

On the other hand, if E is an A-module, the composite $E \to E \otimes_A B \to E \otimes_A C$ is injective (since the pair (A, C) is flat), and so *a fortiori* the same is true of $E \to E \otimes_A B$; this shows that the pair (A, B) satisfies all the hypotheses of Proposition 16.

A.3 Modules over a local ring

In this subsection, A denotes a notherian local ring[21], with maximal ideal $\mathfrak{m}$.

Proposition 18. If a finite type A-module E satisfies the relation $E = \mathfrak{m}E$, the $E = 0$.

[21] Serre: in fact, all the results of the next two sections are valid without change for a *Zariski ring*, see [Sam53b, p. 157].

(See [Sam53b, p.138], or [Car, 1955-6, exp. 1] for example.)

Suppose that $E \neq 0$, and let $e_1, \ldots, e_n$ be a system of generators for E with n minimal. Since $e_n \in \mathfrak{m}E$, we have $e_n = x_1 e_1 + \cdots + x_n e_n$, with $x_i \in \mathfrak{m}$, from which we obtain:

$$(1 - x_n)e_n = x_1 e_1 + \cdots + x_{n-1} e_{n-1};$$

since $1 - x_n$ is invertible in A, this shows that $e_1, \ldots, e_{n-1}$, contradicting the hypothesis on n.

Corollary 5. Let E be an A-module of finite-type. If a submodule F of E satisfies the relation $E = F + \mathfrak{m}E$, we have $E = F$.

The relation implies that $E/F = \mathfrak{m}(E/F)$.

We give every A-module E the $\mathfrak{m}$-adic topology, where the submodules $\mathfrak{m}^n E$ form a neighbourhood base of 0 (see [Sam53b, p. 153]).

Proposition 19. Let E be an A-module of finite type. Then:

a) THe induced topology on a submodule F of E by the $\mathfrak{m}$-adic topology on E coincides with the $\mathfrak{m}$-adic topology on F.

b) Every submodule of E is closed in the $\mathfrak{m}$-adic topology on E (and, in particular, E is Hausdorff).

(See [Sam53b, *loc. cit.*] or [Car, exp. 8].)

Let us speak briefly to the proof of this proposition. We start with a), using the theory of primary decompositions (Krull, see [Sam53b]), which establishes the existence of an integer r such that

$$F \cap \mathfrak{m}^n E = \mathfrak{m}^{n-r}(F \cap \mathfrak{m}^r E) \quad \text{for } n \geq r$$

(Artin-Rees, see [Car, 1955-6, exp. 2]).

We now show that E is Hausdorff: applying a) to the closure of 0, which is a submodule F, we see that $F = \mathfrak{m}F$, so $F = 0$ by Proposition 18. Applying the result to quotient modules of E, we deduce b).

As before, let E be a finite-type A-module, and let $\hat{E}$ and $\hat{A}$ be the completions of E and A for the $\mathfrak{m}$-adic topology. The bilinear map $A \times E \to E$ extends by continuity to a map $\hat{A} \times \hat{E} \to \hat{E}$, which makes $\hat{E}$ an $\hat{A}$-module. As such, the canonical injection of E into $\hat{E}$ extends by linearity to a homomorphism

$$\epsilon : E \otimes_A \hat{A} \longrightarrow \hat{E}.$$

Proposition 20. For every finite-type A-module E, the homomorphism ϵ is bijective.

Let $0 \to R \to L \to E \to 0$ be an exact sequence of A-modules, with L free and of finite type. Since A is noetherian, R is of finite type; on the other hand, Proposition 19 demonstrates that the $\mathfrak{m}$-adic topology on R is induced by that of L, and it is clear that the topology on E is the quotient of that of L. Since the topologies are *metrisable*[22], we deduce from this an exact sequence:

$$0 \longrightarrow \hat{R} \longrightarrow \hat{L} \longrightarrow \hat{E} \longrightarrow 0.$$

Consider now the commutative diagram:

$$\begin{array}{ccccccc}
R \otimes_A \hat{A} & \longrightarrow & L \otimes_A \hat{A} & \longrightarrow & E \otimes_A \hat{A} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \epsilon''} & & \downarrow{\scriptstyle \epsilon'} & & \downarrow{\scriptstyle \epsilon} & & \\
\hat{R} & \longrightarrow & \hat{L} & \longrightarrow & \hat{E} & \longrightarrow & 0
\end{array}$$

The two lines of this diagram are exact sequences, and it is clear that ϵ' is bijective. One deduces that ϵ is surjective (in other words, we have that $\hat{E} = \hat{A} \cdot E$, see [Sam53b, p. 153, Lemma 1]). This result, being demonstrated for any finite-type A-module, applies in particular to R, which shows that ϵ'' is surjective. Applying the Five lemma, we conclude that ϵ is bijective.

[22] A metric is $d(x, y) = 2^{-n}$ for $x \neq y$, where n is such that $x - y \in \mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$.

A.4 Flatness for local rings

All the local rings considered below are supposed to be notherian.

Proposition 21. Let A be a local ring, and let $\hat{A}$ be its completion. The pair $(A, \hat{A})$ is flat.

First, $\hat{A}$ is A-flat. To show this, it suffices to demonstrate that, if $E \to F$ is injective, the same is true of $E \otimes_A \hat{A} \to F \otimes_A \hat{A}$, and we may suppose that E and F are of finite type. In this case, Proposition 20 shows that $E \otimes_A \hat{A}$ is identified with $\hat{E}$, and $F \otimes_A \hat{A}$ with $\hat{F}$, and our assertion results from the evident fact that $\hat{E}$ embeds into $\hat{F}$.

In the same way, the fact that $E \to \hat{E}$ is injective if E is of finite type shows that the pair $(A, \hat{A})$ verifies the property a') of Proposition 16, so forms a flat pair.

Now let A and B be a two local rings, and let $\theta : A \to B$ be a homomorphism. Suppose that θ maps the maximal ideal of A into that of B. Then θ is continuous, and so extends to a homomorphism $\hat{\theta} : \hat{A} \to \hat{B}$.

Proposition 22. Suppose that $\hat{\theta} : \hat{A} \to \hat{B}$ is bijective, and identify A with a subring of B via θ. Then (A, B) is a flat pair.

We have $A \subset B \subset \hat{B} = \hat{A}$, and the pairs $(A, \hat{A})$ and $(B, \hat{B})$ are flat, by the previous proposition. Proposition 17 implies that (A, B) is a flat pair.

Proposition 23. Let A and B be local rings, $\mathfrak{a}$ an ideal of A, and let $\theta : A \to B$ be a homomorphism. If θ satisfies the hypotheses of Proposition 22, the same is true of the induced map $\theta : A/\mathfrak{a} \to B/\theta(\mathfrak{a})B$ (which shows that $(A/\mathfrak{a}, B/\theta(\mathfrak{a})B)$ is a flat pair).

By Proposition 20, the completion of $A/\mathfrak{a}$ is $\hat{A}/\mathfrak{a}\hat{A}$, and similarly that of $B/\theta(\mathfrak{a})B$ is $\hat{B}/\theta(\mathfrak{a})\hat{B}$, which implies the result.

Proposition 24. Let A and A' be two local rings, $\theta : A \to A'$ a homomorphism satisfying the hypotheses of Proposition 22, and let E be an A-module of finite type. If the A'-module $E' = E \otimes_A A'$ is isomorphic to $(A')^n$, then E is isomorphic to A^n.

We identify A with a subring of A' via θ. If $\mathfrak{m}$ and $\mathfrak{m}'$ denote the maximal ideals of A and A', we then have $\mathfrak{m} \subset \mathfrak{m}'$; on the other hand, since $\mathfrak{m}'$ is a neighbourhood of 0 in A', and since A is dense in A', we have $A' = \mathfrak{m}' + A$, which shows that $A/\mathfrak{m} = A'/\mathfrak{m}'$, and so $E/\mathfrak{m}E = E'/\mathfrak{m}'E'$. Since the A'-module E' is a free module of rank n, the same is true of the $A'/\mathfrak{m}'$-module $E'/\mathfrak{m}'E'$. From this, we conclude that it is possible to choose n elements $e_1, \ldots, e_n$ in E so that their images in $E/\mathfrak{m}E$ form a basis for $E/\mathfrak{m}E$, considered as a vector space over $A/\mathfrak{m}$. The elements e_i define a homomorphism $f : A^n \to E$ which is surjective by Corollary 5. We will show that f is injective, which demonstrates the proposition.

Let N be the kernel of f. As the pair (A, A') is flat (Proposition 22), the exact sequence:

$$0 \longrightarrow N \longrightarrow A^n \xrightarrow{f} E \longrightarrow 0,$$

gives rise to an exact sequence:

$$0 \longrightarrow N' \longrightarrow (A')^n \xrightarrow{f'} E' \longrightarrow 0.$$

Since the module E' is free, N' is a direct summand in $(A')^n$, so we have an exact sequence:

$$0 \longrightarrow N'/\mathfrak{m}'N' \longrightarrow (A')^n/\mathfrak{m}'(A')^n \longrightarrow E'/\mathfrak{m}'E' \longrightarrow 0.$$

However, by construction, f' defines a bijection of $(A')^n/\mathfrak{m}'(A')^n$ onto $E'/\mathfrak{m}'E'$. It follows that $N'/\mathfrak{m}'N' = 0$, so $N' = 0$ (Proposition 18), from which it follows that $N = 0$ as the pair (A, A') is flat.

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