Calculating RLCTs

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1 Introduction

The key geometric invariant in Singular Learning Theory is the *Real Log Canonical Threshold* (RLCT) [Wat09]. It can be calculated by resolution of singularities, but in practice this is fiddly. There are some examples, in Watanabe's book and Lin's thesis [Lin11], but I found the process of figuring this out moderately tedious. I will collect here some results and examples that I have come across, most of which will not be original, but will hopefully act as a crash course to make practical calculations easier. The notes [Lin12] are also useful, but focus on the case where the singularity in question is on the interior. Parts are verbatim or paraphrased from my thesis [War21].

Throughout, $W \subset \mathbb{R}^n$ will be a compact semi-analytic set, ie:

$$W = \{\mathbf{x} \mid \psi_1(\mathbf{x}) \geq 0, \dots, \psi_l(\mathbf{x}) \geq 0\},$$

for some analytic functions $\psi_1, \dots, \psi_l$[1]. The ring of real-analytic functions on W is $\mathcal{A}_W$.

Definition 1.1. The RLCT of $f \in \mathcal{A}_W$ on W with the prior φ, is a pair $\mathrm{RLCT}_W(f, \varphi) = (\lambda, \theta)$ defined by either of the following equivalent conditions [Lin11, §4.1.1]

- The log of the "Laplace integral"

  $$\log Z(n) = \log \int_W \exp\left(-n|f(w)|\right)| \varphi(w)|dw,$$

  is asymptotically $-\lambda \log(n) + (\theta - 1) \log\log(n)$ as $n \to \infty$.

- The zeta function

  $$\zeta(z) = \int_W |f(w)|^{-z} |\varphi(w)| dw,$$

  has smallest pole λ, with multiplicity θ.

[1] Strictly speaking, we should assume that W has non-empty interior (see [Wat09, Chapter 6] for the "Fundamental Conditions"). In practice, we will often use the *simplices* $\Delta^m \subset \mathbb{R}^{m+1}$, which do not satisfy this condition — it can however be arranged in the obvious way.

We order pairs $(\lambda_1, \theta_1) < (\lambda_2, \theta_2)$ if, for large enough n:

$$\lambda_1 \log(n) - (\theta_1 - 1) \log\log(n) < \lambda_2 \log(n) - (\theta_2 - 1) \log\log(n).$$

We extend this to ideals of $\mathcal{A}_W$ in the following way. By Lemma 2.3 the value is independent of the generators chosen.

Definition 1.2. The RLCT of an ideal $I = \langle f_1, \ldots, f_r \rangle$ is identified with that of the function $f_1^2 + \cdots + f_r^2$.

Caution: this differs by a factor of 2 from the definition in [Lin11]. As a result:

$$\mathrm{RLCT}_W(f; \varphi) = (\lambda, \theta)$$
$$\mathrm{RLCT}_W(\langle f \rangle; \varphi) = (\lambda/2, \theta).$$

First we collect some useful results for calculations, which are proven in the sequel.

- If $|g| \leq c|f|$ for some constant c, then $\mathrm{RLCT}_W(g; \varphi) \leq \mathrm{RLCT}_W(f; \varphi)$ (Lemma 2.1).
- Analogously perhaps, if $I \subset J$ are ideals, then $\mathrm{RLCT}_W(I; \varphi) \leq \mathrm{RLCT}_W(J; \varphi)$ (Corollary 2.4). Also, for every $r > 0$ (Proposition 2.5):

  $$\mathrm{RLCT}_W(I; \varphi) = (\lambda, \theta) \implies \mathrm{RLCT}_W(I^r; \varphi) = (\lambda/r, \theta).$$

- Suppose that $P \in W \subset \mathbb{R}^n$ lies in the interior, and let $\mathfrak{m}_P$ denote the maximal ideal at P. Then if $\mathrm{ord}_P(I)$ is the largest integer K so that $I \subset \mathfrak{m}^K$, we have (Corollary 2.8):

  $$\mathrm{RLCT}_W(I; 1) \leq \left(\frac{n}{2 \cdot \mathrm{ord}_P(I)}, 1 \right).$$

  For functions, this agrees with the usual order of vanishing at P.

- Let W_1 and W_2 be semi-analytic, and $J_i \subset \mathcal{A}_{W_i}$ ideals. Let $W = W_1 \times W_2$, and by composing with the projections, we consider $\mathcal{A}_{W_i} \subset \mathcal{A}_W$. Suppose $\mathrm{RLCT}_{W_i}(J_i; \varphi_i) = (\lambda_i, \theta_i)$. Then Proposition 2.10 gives us the formulae:

  $$\mathrm{RLCT}_W(J_1 + J_2; \varphi) = (\lambda_1 + \lambda_2, \theta_1 + \theta_2 - 1)$$
  $$\mathrm{RLCT}_W(J_1 J_2; \varphi) = \begin{cases} (\lambda_1, \theta_1) & \lambda_1 < \lambda_2 \\ (\lambda_2, \theta_2) & \lambda_2 < \lambda_1 \\ (\lambda_1, \theta_1 + \theta_2) & \text{else} \end{cases}$$

  where $\varphi = \varphi_1 \times \varphi_2$.

- The primary reason for examining the RLCTs of ideals is that, using the result of Lemma 2.11, we can replace a complicated function with a simpler ideal. See below for the precise statement, but here is a special case. Let Z be a finite set, and ΔZ the standard simplex over Z. If $p : W \to \Delta Z$ is a parametrised family of probability distributions over Z, and $q \in \Delta Z$ is some "true distribution", set

  $$K(w) = D_{\mathrm{KL}}\big(q \mid\mid p(w)\big) := \sum_{z \in Z} q(z) \log\left(\frac{q(z)}{p(z)}\right).$$

  In this case K has the same RLCT as the ideal:

  $$\big\langle p(w)(z) - q(z) \ : \ z \in Z \big\rangle.$$

A more meaty result is the following, which is proved in [Lin11].

Theorem 1.3. Let f and φ be real-analytic on W, and ϕ smooth and strictly positive. Then around every $w \in W$ there is a neighbourhood $N_w \subset W$ so that:

$$\mathrm{RLCT}_{N_w}(f; \varphi\phi) = \mathrm{RLCT}_{N_w}(f; \varphi).$$

Moreover,

$$\mathrm{RLCT}_W(f; \varphi\phi) = \min_{w \in \mathbb{V}(f)} \mathrm{RLCT}_{N_w}(f; \varphi).$$

In what follows, given $w \in W$, N_w will always denote the neighbourhood of the theorem, and by

$$\mathrm{RLCT}_w(f; \varphi),$$

we mean the RLCT calculated on an open neighbourhood in $\mathbb{R}^n$ (ie ignoring the boundary of W). We have that,

$$\mathrm{RLCT}_w(f; \varphi) \leq \mathrm{RLCT}_{N_w}(f; \varphi).$$

This follows as, if $U \subset \mathbb{R}^n$ is our open neighbourhood, we can shrink $N_w \subset U$ so that $Z_{N_w}(n) \leq Z_U(n)$ for every n.

The theorem is proven by reduction to the case where f and φ are monomials, integrated over the positive orthant (Proposition 2.9). This uses *resolution of singularities*, which is deep and difficult — Theorem 3.1. Roughly, this (algorithmic) process takes a possibly singular subvariety X of a smooth variety W, and produces a map $\pi : \tilde{W} \to W$ which is an isomorphism over the complement of the singular locus of X, and which "desingularises" X. Specifically, the *strict transform*:

$$\tilde{X} := \overline{\pi^{-1}(X \setminus \mathrm{Sing}(X))}$$

is smooth (see [Hau14, Lecture 7] for more precise statements of the various forms of this theorem). In Section 3 we sketch a loose understanding of this process, which allows one to calculate RLCTs in practice.

The core of the resolution algorithm is a transform called *blowing up* (again, we sketch this is slightly more detail in Section 3.1). To desingularise $X \subset W$, we pick some smooth subvariety $Z \subset \mathrm{Sing}(X)$, and compute the blowing-up, which is a map from a new variety $\mathrm{Bl}_Z(W) \to W$. (Note that the blowing up is performed on W, but the centre Z is determined by X.) Repeating this process eventually gets us our resolution. The general case is given below, but in every example from SLT that I have seen the centre Z is some coordinate subspace.

Let $W = \mathbb{R}^m \times \mathbb{R}^n$, with coordinates $(x_1, \ldots, x_m, y_1, \ldots, x_n)$, and $Z = \mathbb{R}^m$. Then $\mathrm{Bl}_Z(W)$ has the description:

$$\mathrm{Bl}_Z(W) := \{(x, y, l) \mid x \in l\} \subset \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{P}^{m-1}.$$

With homogenous coordinates $[u_1 : \cdots : u_m]$ on $\mathbb{P}^{m-1}$, the affine piece $u_m \neq 0$ has coordinates $(x_m, y_1, \ldots, y_n, u_1, \ldots, u_{m-1})$, and the map π is:

$$\pi(x_m, y_1, \ldots, y_n, u_1, \ldots, u_{m-1}) = (u_1 x_m, \ldots, u_{m-1} x_m, x_m, y_1, \ldots, y_n).$$

More detail is given below, but this is about as far as [Wat09] goes.

2 Results

Lemma 2.1. Let f, g be analytic functions on W. If there is a constant $c > 0$ such that $|g(w)| \leq c|f(w)|$ for every $w \in W$, then:

$$\mathrm{RLCT}_W(g; \varphi) \leq \mathrm{RLCT}_W(f; \varphi).$$

Proof. Using monotonicity, we have that $Z_g(n) \leq Z_f(cn)$. Asymptotically this gives us:

$$\lambda_g \log(n) - (\theta_g - 1)\log\log(n) \leq \lambda_f \log(cn) - (\theta_f - 1)\log\log(cn) + O(1).$$

This implies the required inequality, as the constant gets absorbed into the constant term. □

Corollary 2.2. If there are constants $c_1, c_2 > 0$ so that $c_1|f(w)| \leq |g(w)| \leq c_2|f(w)|$, then $\mathrm{RLCT}_W(f;\varphi) = \mathrm{RLCT}_W(g;\varphi)$. Such functions are called *comparable*.

Lemma 2.3. Let $f_1, \dots, f_r$ and $g_1, \dots, g_s$ be analytic on $\mathcal{W}$. Then if $< g_1, \dots, g_s > \subseteq < f_1, \dots, f_r >$, then

$$\mathrm{RLCT}_W(g_1^2 + \dots, g_s^2;\varphi) \leq \mathrm{RLCT}_W(f_1^2 + \dots + f_r^2;\varphi).$$

Proof. For each $i = 1, \dots, s$, we can find analytic functions $h_1, \dots, h_r$ on W so that:

$$g_i = h_1 f_1 + \dots + h_r f_r.$$

Using the Cauchy-Schwartz inequality, and the fact that W is compact:

$$g_i^2 = \left(\sum_j h_j f_j\right)^2 \leq \left(\sum_j h_j^2\right)\left(\sum_j f_j^2\right) \leq c_i \left(\sum_j f_j^2\right).$$

For some constant c_i. This implies that

$$\sum_i g_i^2 \leq \left(\sum_i c_i\right)\left(\sum_j f_j^2\right),$$

which by Lemma 2.1 implies the result. □

Corollary 2.4. If $I \subset J$ are ideals of A_W, then:

$$\mathrm{RLCT}_W(I;\varphi) \leq \mathrm{RLCT}_W(J;\varphi).$$

Proposition 2.5. Let $I \subset \mathcal{A}_W$ be an ideal, and $\mathrm{RLCT}_W(I;\varphi) = (\lambda, \theta)$. Then:

$$\mathrm{RLCT}_W(I^r;\varphi) = (\lambda/r, \theta).$$

Proof. It is obvious from the definition in terms of zeta-functions that, for an analytic function f, $\mathrm{RLCT}_W(f^r;\varphi) = (\lambda/r, \theta)$ when $\mathrm{RLCT}_W(f;\varphi) = (\lambda, \theta)$. Let the sum-of-squared-generators associated to I^r be $f^{(r)}$. By the previous observation, it suffices to demonstrate that

$$f^{(r)} \text{ is comparable to } \left(f^{(1)}\right)^r.$$

Letting $I = \langle f_1, \dots, f_k \rangle$, and $\mathbf{e} \in \mathbb{N}^k$ range over multi-indexes, we have:

$$f^{(r)} = \sum_{|\mathbf{e}|=r} \mathbf{f}^{2\mathbf{e}}$$

$$\left(f^{(1)}\right)^r = \sum_{|\mathbf{e}|=r} \binom{r}{\mathbf{e}} \mathbf{f}^{2\mathbf{e}}.$$

In the previous, we use the notations for a multi-index $\mathbf{e} = (e_1, \ldots, e_k)$:

$$|\mathbf{e}| = \sum_{i=1}^{k} e_i$$
$$\mathbf{f}^{2\mathbf{e}} = f_1^{2e_1} \cdots f_k^{2e_k}$$
$$\binom{r}{\mathbf{e}} = \frac{r!}{e_1! \cdots e_k!}$$

Since

$$\sum_{|\mathbf{e}|=r} \binom{r}{\mathbf{e}} = (1 + \cdots + 1)^r = k^r,$$

we have:

$$f^{(r)} \leq \left(f^{(1)}\right)^r \leq k^r f^{(r)},$$

which completes the proof. □

We next observe two simple cases.

Proposition 2.6. Let $m_1, \ldots, m_n$ be positive integers, and $f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^{m_i}$. Then if $\mathbb{R}^n_{\geq 0}$ denotes the positive orthant:

$$\mathrm{RLCT}_{\mathbb{R}^n_{\geq 0}}(f; 1) = \left(\sum_{i=1}^{n} \frac{1}{m_i}, 1\right).$$

Proof. Calculating the Lapace integral for f:

$$\begin{aligned}
Z(n) &= \int_{\mathbb{R}^n_{\geq 0}} e^{-nf(\mathbf{x})} d\mathbf{x} \\
&= \prod_{i=1}^{n} \int_0^{\infty} e^{-nx^{m_i}} dx \\
&= \prod_{i=1}^{n} \int_0^{\infty} e^{-u_i^{m_i}} \frac{du_i}{n^{1/m_i}} \\
&= \prod_{i=1} \text{const} \cdot n^{-1/m_i} \\
&= \text{const} \cdot n^{-\sum_i \frac{1}{m_i}}.
\end{aligned}$$

□

Corollary 2.7. Set $\mathfrak{m} = (x_1, \ldots, x_n)$. Then

$$\mathrm{RLCT}_0(\mathfrak{m}; 1) = \left(\frac{n}{2}, 1\right).$$

Corollary 2.8. Let $K = \mathrm{ord}_0(I)$ be the largest natural number so that $I \subset \mathfrak{m}^K$. Then

$$\mathrm{RLCT}_0(I; 1) \leq \left(\frac{n}{2 \cdot \mathrm{ord}_0(I)}, 1\right).$$

Proof. By Proposition 2.5 and Corollaries 2.4 and 2.7. □

Proposition 2.9. Let $k = (k_1, \ldots, k_n)$ and $h = (h_1, \ldots, h_n)$ be vectors of non-negative integers, and ϕ a smooth function of compact support, with $\phi(0) > 0$. Then if $\mathrm{RLCT}_{\mathbb{R}^n_{\geq 0}}(x^k; x^h\phi) = (\lambda, \theta)$, we have:

$$\lambda = \min_i \left\{ \frac{h_i + 1}{k_i} \right\},$$

and θ is the number of i for which this minimum is attained.

Proof. See [Lin11, Proposition 3.7], with more detail in [AVGZ85, Lemma 7.3]. For $\phi(x) = 1$, we can integrate our zeta function explicitly (taking $x \in [0, K]^d$ as ϕ is in fact compactly supported):

$$\begin{aligned}
\zeta(z) &= \int_W dx\, x^{\tau - z\kappa} \\
&= \prod_{i=1}^{d} \int_0^K dx\, x^{\tau_i - z\kappa_i} \\
&= \prod_{i=1}^{d} \frac{K^{1+\tau_i - z\kappa_i}}{1 + \tau_i - z\kappa_i}.
\end{aligned}$$

In this situation we have poles for $1 + \tau_i - z\kappa_i = 0$, so the statement is clear. The general case follows by expanding ϕ into an N^{th} order Taylor series and remainder. The (non-zero) constant term contributes the smallest pole, and by increasing N, the term involving the remainder can be made analytic. □

Proposition 2.10. Let W_1 and W_2 be semi-analytic, and $J_i \subset \mathcal{A}_{W_i}$ ideals. Let $W = W_1 \times W_2$, and by composing with the projections, we consider $\mathcal{A}_{W_i} \subset \mathcal{A}_W$. Suppose $\mathrm{RLCT}_{W_i}(J_i; \varphi_i) = (\lambda_i, \theta_i)$. Then:

$$\mathrm{RLCT}_W(J_1 + J_2; \varphi) = (\lambda_1 + \lambda_2, \theta_1 + \theta_2 - 1)$$

$$\mathrm{RLCT}_W(J_1 J_2; \varphi) = \begin{cases} (\lambda_1, \theta_1) & \lambda_1 < \lambda_2 \\ (\lambda_2, \theta_2) & \lambda_2 < \lambda_1 \\ (\lambda_1, \theta_1 + \theta_2) & \text{else} \end{cases}$$

We set $\varphi = (\varphi_1 \circ \pi_1)(\varphi_2 \circ \pi_2)$.

Proof. Let f_i be the function defining $\mathrm{RLCT}_{W_i}(J_i; \varphi_i)$. For the first equality, examine the Laplace integral:

$$\begin{aligned}
Z(n) &= \int_W e^{-n(f_1(x_1) + f_2(x_2))} \varphi_1(x_1)\varphi_2(x_2) dx_1 dx_2 \\
&= \left(\int_{W_1} e^{-nf_1(x_1)} \varphi_1(x_1) dx_1 \right) \left(\int_{W_2} e^{-nf_2(x_2)} \varphi_2(x_2) dx_2 \right) \\
&\sim \left(C_1 n^{-\lambda_1} (\log n)^{\theta_1 - 1} \right) \left(C_2 n^{-\lambda_2} (\log n)^{\theta_2 - 1} \right) \\
&= C n^{-\lambda_1 - \lambda_2} (\log n)^{\theta_1 + \theta_2 - 2}.
\end{aligned}$$

In the same way, if $\zeta(x_1, x_2)$ is the zeta function associated to $J_1 J_2$, we have that $\zeta(x_1, x_2) = \zeta_1(x_1)\zeta_2(x_2)$. Therefore, it has its smallest pole at $\min\{\lambda_1, \lambda_2\}$. If these coincide, their multiplicities add. □

Lemma 2.11. Suppose that $\mathcal{W} \subset \mathbb{R}^d$ and $\mathcal{W}' \subset \mathbb{R}^{d'}$ are compact and semi-analytic, and $f = (f_1, \ldots, f_{d'}) : \mathcal{W} \to \mathcal{W}'$ and $g : \mathcal{W}' \to \mathbb{R}$ are real analytic. Pick $\hat{w} \in \mathcal{W}$, set $\hat{f} = f(\hat{w})$. Then if $g(\hat{f}) = 0$, $\nabla g(\hat{f}) = 0$ and the Hessian $\nabla^2 g(\hat{f})$ is positive definite, then:

$$\mathrm{RLCT}_{N_{\hat{w}}}(g \circ f; \varphi) = \mathrm{RLCT}_{N_{\hat{w}}}(\langle f_1 - \hat{f}_1, \ldots, f_{d'} - \hat{f}_{d'} \rangle; \varphi).$$

Proof. See [Lin11, Proposition 4.4]. The lemma follows from the fact that, in a small enough neighbourhood of $\hat{f}$, g is comparable (in the sense of Corollary 2.2) to a sum of squares:

$$(u_1 - \hat{f}_1)^2 + \cdots + (u_{d'} - \hat{f}_{d'})^2,$$

where $u_1, \ldots, u_{d'}$ are coordinates on $\mathcal{W}'$. The right-hand side is exactly the RLCT of f composed with such a sum of squares. □

3 Resolution of Singularities

The result we use is the following, which sometimes goes by the name "local monomialisation". We follow Watanabe and Lin in using a version due to Atiyah [Ati70].

Theorem 3.1 (Resolution of Singularities)**.** Let f be a non-constant real analytic function on a neighbourhood of the origin in $\mathbb{R}^d$, with $f(0) = 0$. Then there exists a triple (M, W, ρ) where:

- $W \subset \mathbb{R}^d$ is open, and contains 0,
- M is a d-dimensional real analytic manifold,
- $\rho : M \to W$ is a real analytic map.

The following also hold.

- ρ is proper, the inverse image of a compact set is compact.
- ρ is a real analytic isomorphism away from $\mathbb{V}_W(f)$. (That is, $M \setminus \mathbb{V}_M(f \circ \rho) \longrightarrow W \setminus \mathbb{V}_W(f)$.)
- Around any point $y \in \mathbb{V}_M(f \circ \rho)$, there are local coordinates $u = (u_1, \ldots, u_d)$ on some neighbourhood M_y, vectors κ and τ of non-negative integers, and strictly positive, real analytic functions a and h of u such that:
$$f \circ \rho(u) = a(u)u^{\kappa},$$
and the Jacobian determinant of ρ:
$$|\rho'(u)| = h(u)u^{\tau}.$$

Corollary 3.2. Given non-constant analytic functions $f_1, \ldots, f_l$ in a neighbourhood of $0 \in \mathbb{R}^d$, all vanishing at the origin, there is a triple (M, W, ρ) as above which desingularises each f_i.

Proof. See [Wat09, Theorem 2.8]. Apply the original form of the Theorem to the product $f_1(w) \cdots f_l(w)$, then observe [Wat09, Theorem 2.7] that the resulting triple desingularises each f_i. □

Now we want to apply this theorem to calculate RLCTs: in short, it works as follows [Lin11, Lemma 3.8]. The statement is local, so we examine a particular point $w \in \mathbb{V}(f)$, and desingularise f at w, using the theorem. We may also assume that the triple (M, W, ρ) desingularises φ and each of the analytic functions $\psi_1, \ldots, \psi_l$ (if they vanish at w) which define $W \subset \mathbb{R}^d$. We can also show that the neighbourhood W can be shrunk to N_w such that $\rho^{-1}(N_w)$ is a union of coordinate neighbourhoods M_y as in the theorem. In each of these coordinates, the situation is as in Proposition 2.9: since the constraints are monomial, $\mathcal{M}_y = M_y \cap \rho^{-1}\mathcal{W}$ is a union of orthants, and the functions $f \circ \rho, \varphi \circ \rho$ are of the correct form. Using a partition of unity $\{\sigma_y\}$ subordinate to $\{\mathcal{M}_y\}$, we can write the zeta function as:

$$\zeta(z) = \sum_y \int_{\mathcal{M}_y} du\, |f \circ \rho(u)|^{-z} |\varphi \circ \rho(u)| |\phi \circ \rho(u)| \sigma_y(u).$$

The RLCT (λ, θ) associated to ζ is simply the smallest such pair (using the usual ordering) associated to one of the integrals:

$$\zeta_y(z) = \int_{\mathcal{M}_y} du\, |f \circ \rho(u)|^{-z} |\varphi \circ \rho(u)||\phi \circ \rho(u)| \sigma_y(u),$$

which we may calculate as in the proposition.

The manifold M and map ρ are computed by transformations called blow-ups. The precise algorithm is beyond the scope of these notes, but the following subsection sketches how to find such a map by trial and error.

3.1 Blowing up

The abstract content of a blow up — framed here inside scheme theory — may be summarised by the following. The condition "$\tilde{\mathcal{J}}$ is invertible" may be replaced with "$\tilde{\mathcal{J}}$ is locally generated by a single non-zero-divisor" in the case that X is an integral scheme (eg a variety).

Proposition ([Mur05], Proposition 3)**.** Let X be a noetherian scheme, $\mathcal{J}$ a coherent sheaf of ideals, and let $\pi : \tilde{X} \to X$ be the blowing up of $\mathcal{J}$. Then

- The inverse image ideal sheaf $\tilde{\mathcal{J}} := \mathcal{J} \cdot \mathcal{O}_{\tilde{X}}$ is an invertible sheaf on $\tilde{X}$.
- If Y is the closed subset corresponding to $\mathcal{J}$, then $\pi^{-1}U \to U$ is an isomorphism, where $U = X \setminus Y$.

Theorem ([Mur05], Theorem 9)**.** Let X be a noetherian scheme, $\mathcal{J}$ a coherent sheaf of ideals, and $\pi : \tilde{X} \to X$ the blowing-up of X with respect to $\mathcal{J}$. If $f : Z \to X$ is any morphism such that $\mathcal{J} \cdot \mathcal{O}_Z$ is an invertible sheaf of ideals on Z, then there exists a unique morphism $g : Z \to \tilde{X}$ making the following diagram commute

$$\begin{array}{ccc} Z & \overset{g}{\dashrightarrow} & \tilde{X} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} \\ & & X. \end{array}$$

The monomialisation of Theorem 3.1 is achieved by a sequence of blow ups of the *ambient* space $\mathbb{R}^d$ — the ideal sheaf $\mathcal{J}$ is chosen to cut out a smooth subvariety V of $\mathbb{R}^d$, and the subvariety we chose is dictated by the singularities of f. We don't need to dive into the full generality of schemes and such, and in fact most of the blow ups I have come across have simply been of a coordinate axis. Let us examine these formulae. For a more rigorous and detailed accounts, see notes [Smi, Hau14] or [Spi20].

Definition 3.3. Let $X \subset \mathbb{R}^n$ be open, and $I = (f_1, \dots, f_r)$ an ideal in $\mathcal{A}_X$. Define a map:

$$\phi : X \setminus \mathbb{V}(I) \longrightarrow \mathbb{P}^{r-1},$$

by $\phi(x) = [f_1(x) : \cdots : f_r(x)]$ in homogenous coordinates. Then the blowing-up $\tilde{X}$ is the closure of the graph of ϕ, ie, the set:

$$\tilde{X} = \overline{\{(x, [f_1(x) : \cdots : f_r(x)]) \mid x \in X \setminus \mathbb{V}(I)\}} \subset X \times \mathbb{P}^{r-1}.$$

The map $\pi : \tilde{X} \to X$ is the projection onto the first coordinate.

Let us examine in detail the case where $X = \mathbb{R}^d$ and $I = (x_1, \dots, x_d)$, so $\mathbb{V}(I)$ is the origin. Then, identifying $\mathbb{P}^{d-1}$ with lines through the origin in $\mathbb{R}^d$, $\tilde{X}$ has the description:

$$\tilde{X} = \{(x, l) \mid x \in l\} \subset \mathbb{R}^d \times \mathbb{P}^{d-1}.$$

If $x \neq 0$, it defines a unique line $l \in \mathbb{P}^{d-1}$ such that $x \in l$ — this is why π is an isomorphism away from the preimage of the origin. Since $0 \in l$ for every $l \in \mathbb{P}^{d-1}$, $E := \pi^{-1}(0) = \mathbb{P}^{d-1}$. To see why this has a

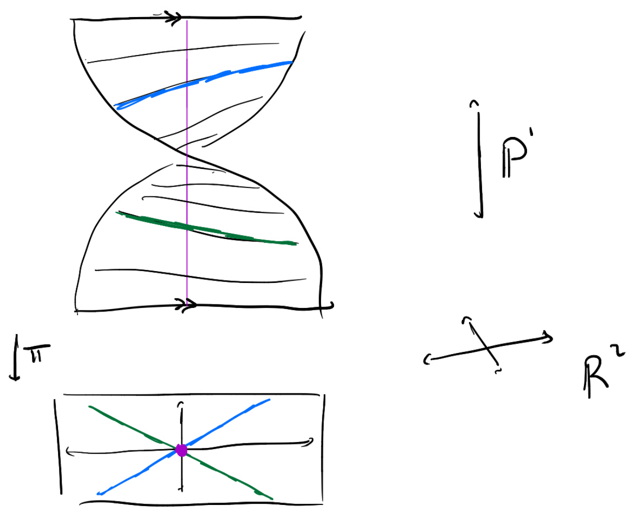

Figure 1: The blowing-up of $\mathbb{R}^2$ with centre $(0,0)$, two lines and their preimages, as well as the exceptional locus E, are marked. Note that the top and bottom and glued together, so that $\tilde{X}$ is topologically an open Möbius strip.

chance of desingularising some subvariety, first observe that for two distinct lines $l, l' \subset \mathbb{R}^d$, the inverse images $\pi^{-1}(l \setminus 0)$ and $\pi^{-1}(l' \setminus 0)$ approach E in different places. Specifically, they limit towards $(0, l)$ and $(0, l')$. With $d = 2$ we can visualise this as in Figure 1.

A possible singularity at zero is a node, take $f = y^2 - x^2(x+1)$ for example, which crosses twice through the origin with slopes ± 1. The *strict transform* of the singular variety $Y = \mathbb{V}(f)$ is

$$\tilde{Y} := \overline{\pi^{-1}(Y \setminus \{0\})},$$

(so called to differentiate it from the *total transform* $\pi^{-1}(Y)$). This splits apart the two crossings, as they limit towards two different points in E (see Figure 2).

We can give a more helpful description as follows. Let $[u_1 : \cdots : u_d]$ be homogenous coordinates on $\mathbb{P}^{d-1}$, and observe that the point $(x_1, \ldots, x_d)$ lies in the line so defined if and only if:

$$x_i u_j - x_j u_i = 0, \quad \forall i, j = 1, \ldots, d.$$

Therefore, the blowing-up $\tilde{X}$ of the origin in $\mathbb{R}^d$ may be expressed as the vanishing locus:

$$\tilde{X} = \mathbb{V}(x_i u_j - x_j u_i \,:\, i, j = 1, \ldots, d) \subset \mathbb{R}^d \times \mathbb{P}^{d-1}. \tag{1}$$

This gets us somewhere if we consider the affine charts on $\mathbb{P}^{d-1}$ (in the latter expression the term u_i/u_i is skipped):

$$U_i = \mathbb{R}^d \times \{u_1 : \cdots : u_d] \mid u_i \neq 0\} \cong \mathbb{R}^d \times \left\{ \left(\frac{u_1}{u_i}, \ldots, \frac{u_d}{u_i} \right) \right\} \subset \mathbb{R}^{2d-1}.$$

Then, $\tilde{X} \cap U_i \cong \mathbb{R}^d$ with coordinates $(t_1^{(i)}, \ldots, t_d^{(i)})$ given by:

$$t_j^{(i)} = \begin{cases} x_i & i = j \\ u_j / u_i & i \neq j. \end{cases}$$

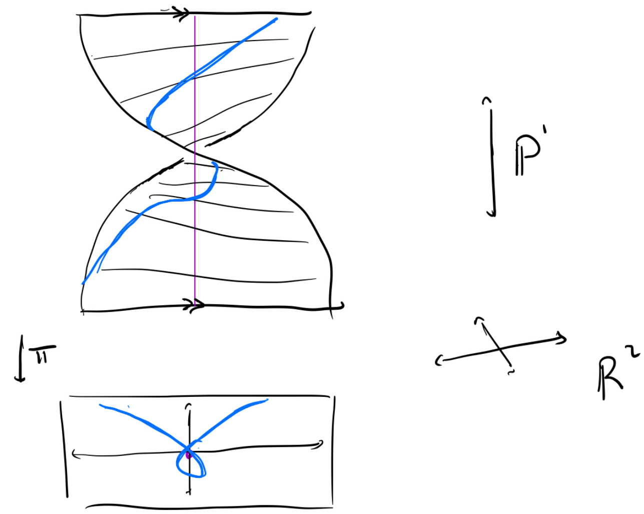

Figure 2: The strict transform of the nodal cubic.

Solving for x_j, the map π has the expression:

$$\pi(t_1^{(i)}, \dots, t_d^{(i)}) = (t_1^{(i)} t_i^{(i)}, \dots, t_i^{(i)}, \dots, t_d^{(i)} t_i^{(i)}).$$

This is, usually, all that one needs: see for example [Lin12, §9] or [War21, Example 3.23]. This example extends easily to the case $I = (x_1, \dots, x_r)$ for $r < d$ — set $t_j^{(i)} = x_j$ for $j > r$.

We can extend eq. (1) to the general case. Let $I = (f_1, \dots, f_r)$, so that our equations become:

$$f_i(x)u_j - f_j(x)u_i = 0, \quad i, j = 1, \dots, r.$$

Observe that this defines a subset of $\mathbb{R}^d \times \mathbb{P}^{r-1}$ as each equation is homogenous in the u_i. The charts are (away from the exceptional locus):

$$U_i \cap \Gamma_\phi = \{(x, \phi(x)) \mid f_i(x) \neq 0\}. \tag{2}$$

A nice way of expressing this is the following ([Spi20, §1.3]): let $A = \mathbb{R}[X_1, \dots, X_d]$ be the coordinate ring of $\mathbb{R}^d$, then the affine piece $\tilde{X} \cap U_i$ has associated ring (where Y_i is skipped):

$$\frac{A[Y_1, \dots, Y_r]}{\big(f_i(X)Y_j - f_j(X)\big)} = A\left[\frac{f_1(X)}{f_i(X)}, \dots, \frac{f_r(X)}{f_i(X)}\right].$$

The expression on the left only works if the $f_1, \dots, f_r$ form a *regular sequence.* That is, if for every $m < r$ g_m is a non-zero-divisor in $A/(f_1, \dots, f_{m-1})$, meaning the fractions f_i/f_j do not satisfy any non-trivial linear relations over A. For example ([Hau14, Example 4.43]), if $I = (x^2, xy, y^3) \subset A = K[x, y]$, then

$$A\left[\frac{xy}{x^2}, \frac{y^3}{x^2}\right] \cong A\left[\frac{x}{y}\right],$$

which is different to

$$\frac{A[u, v]}{(x^2u - xy, x^2v - y^3)}.$$

Namely, we have to add the equation $u^2y - v$. This corresponds to the fact that the closure of the graph in eq. (2) is smaller than expected, due to the extra linear relations satisfied by the generators of I.

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